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Noetherian spaces, wqos, and their statures

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Outline

❖ Wqos, WSTS… your five minutes of comfort
❖ Extending classical theorems from wqos to Noetherian spaces
❖ Sobrifications of Noetherian spaces, and their representations
❖ Statures of Noetherian spaces and maximal order types of wqos
Wqos and WSTS
**Fact.** The following are equivalent for a quasi-ordering \( \leq \):

1. Every sequence \( (x_n)_{n \in \mathbb{N}} \) is **good**: \( x_m \leq x_n \) for some \( m < n \)
2. Every sequence \( (x_n)_{n \in \mathbb{N}} \) is **perfect**: has a monotone subsequence
3. \( \leq \) is **well-founded** and has no infinite antichain.
4. Every **upwards-closed** subset is the upwards-closure \( \uparrow \{x_1, \ldots, x_n\} \) of a **finite** set
5. Every monotonic chain \( U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots \) of upwards-closed subsets is **stationary** (i.e., all the sets \( U_n \) are equal from some rank on)

**Defn.** Such a quasi-ordering \( \leq \) is called a **well-quasi-order** (wqo).
Examples

- \(\mathbb{N}\), with its usual ordering  — More generally, any total well-founded order
- Every finite set, with any quasi-ordering
- Finite disjoint sums, finite products of wqos are wqo
- Images of wqos by monotonic maps are wqo (in particular quotients)
- Inverse images of wqos by order-reflecting maps are wqo (in particular subsets)
- Higman’s Lemma. Let \(X^* = \{\text{finite words over alphabet } X\}\) ordered by word embedding \(\leq^*\). Then \(X\) wqo \(\iff\) \(X^*\) wqo
- Kruskal’s Theorem. Let \(\mathcal{T}(X) = \{\text{finite trees with } X\text{-labeled vertices}\}\) ordered by homeomorphic tree embedding \(\leq_T\). Then \(X\) wqo \(\iff\) \(\mathcal{T}(X)\) wqo.
- And so on.
Well-structured transition systems

* A very interesting class of (infinite) transition systems where coverability (a special form of reachability) is **decidable**

**Definition.** A well-structured transition system (WSTS) is a transition system \((X, \rightarrow)\) with a \(\text{wqo} \leq\) on \(X\) satisfying (strong) monotonicity:

\[ \forall x \exists y (\ni x' \ni x \ni y' \ni y) \]

... and many other examples

letters can spontaneously vanish from communication queues (needed for decidability... and rather realistic)
Coverability is decidable

**Effective WSTSs:**
- points are representable
- $\leq$ is decidable
- $y \mapsto \{x_1, \ldots, x_n\} = \text{Pre}(\uparrow y)$ is computable (so one can compute $\text{Pre}(U)$)

**Theorem.** (Abdulla et al. 2000, Finkel&Schnoebelen 2001.) Coverability is **decidable** on effective WSTSs.

\begin{verbatim}
fun pre* U = 
  let V = pre U 
  in 
  if V \subseteq U 
    then U 
    else pre* (U \cup V) 
  end; 

fun coverability (s, B) = 
  s in pre* (B); 
\end{verbatim}

**Definition.** A well-structured transition system (WSTS) is a transition system $(X, \rightarrow)$ with a wqo $\leq$ on $X$ satisfying (strong) monotonicity:

\[
\forall x, y 
\exists x' 
\forall y'.
\]
Beyond wqos: Noetherian spaces
Going topological

- Every quasi-ordered set \((X, \leq)\) gives rise to a topological space, whose open sets are the upwards-closed sets (the Alexandroff topology).

- **Definition.** A topological space is **Noetherian** iff every monotonic chain \(U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots\) of open subsets is **stationary**.

- **Proposition.** \((X, \leq)\) is wqo iff \(X\) is Noetherian in its Alexandroff topology.

- Hence Noetherian spaces generalize wqos.
Is the generalization proper?

- **Yes.** Consider \( \mathbb{N}_{\text{cof}} \), the set of natural numbers with the **cofinite topology**, whose closed sets are the finite subsets (plus \( \mathbb{N} \))

- It may be easier to see that \( \mathbb{N}_{\text{cof}} \) is Noetherian by realizing that:

  - **Proposition.** A space \( X \) is Noetherian iff 
    every **antitonic** chain \( F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots \) of **closed** subsets is **stationary**.
    
    (Take complements.)

- \( \mathbb{N}_{\text{cof}} \) does **not** arise from a wqo, because its specialization ordering is =, which is never wqo on an infinite set.
Properties T and W

- Let \((X, \leq)\) be a quasi-ordered set. Its \textit{finitary} subsets are \(\downarrow \{x_1, \ldots, x_n\}\).

- The finitary subsets generate the \textit{upper topology}. It, too, has \(\leq\) as specialization quasi-ordering.

\[ \text{The upper topology is the coarsest topology with } \leq \text{ as specialization.} \]

The Alexandroff topology is the \textit{finest}.

\[ \text{This turns out to be the general form of all sober Noetherian spaces.} \]

\[ \text{Properties T and W} \]

- **Proposition.** If:
  - \(X\) is \textit{well-founded}
  - (Property T) \(X\) is \textit{finitary}
  - (Property W) For all \(x, y \in X\), \(\downarrow x \cap \downarrow y\) is \textit{finitary}

  then \(X\) is \textit{Noetherian} in the upper topology and the \textit{closed} sets are the \textit{finitary} subsets.

\[ \text{Noetherian} \]
Well-founded trees

- **Every well-founded tree** (even not finitely branching) is **Noetherian** in the upper topology

- (Property T) \( X = \downarrow \{ r_0 \} \)

- (Property W) \( \downarrow x \cap \downarrow y \) is empty or equal to \( \downarrow x \) or \( \downarrow y \)

- **Not wqo** unless tree is finitely branching (infinite antichains)

- Closed sets = finite disjoint unions of **subtrees**

**Proposition.** If:

- \( X \) is well-founded
- (Property T) \( X \) is finitary
- (Property W) For all \( x, y \in X \), \( \downarrow x \cap \downarrow y \) is finitary

then \( X \) is **Noetherian** in the upper topology.
The Hoare hyperspace of a Noetherian space

- Every **well-founded inf-semilattice** is Noetherian in the upper topology

- Let $\mathcal{H}X = \{\text{closed subsets of } X\}$ with the upper topology of $\subseteq$
  (Hoare hyperspace of $X$)

- $\mathcal{H}(X)$ is an inf-semilattice, hence:

  **Proposition.** If $X$ is Noetherian, then $\mathcal{H}X$ is Noetherian.

  (That is actually an equivalence.)

X is Noetherian iff:
- (5) Every monotonic chain $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots$ of open subsets is **stationary**
- (6) Every antitonic chain $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ of closed subsets is **stationary**
- (7) $\mathcal{H}X$ is well-founded.
Finite words

- Let \( X^* = \{ \text{finite words on } X \} \) with word topology:
  basic open sets \( \langle U_1, \ldots, U_n \rangle = X^* U_1 X^* \cdots X^* U_n X^* \), where each \( U_i \) is open in \( X \)
- The logical view: words as specific finite structures,
  word topology = \textbf{disjunctive queries} \hspace{1cm} (in database parlance)
  \[ F ::= i \in U \mid i < j \mid \bot \mid \top \mid F \lor F \mid F \land F \mid \exists i, F \quad (U \text{ open in } X) \]
- \( \langle U_1, \ldots, U_n \rangle = \{ \text{finite words satisfying } \exists i_1, \ldots, i_n \cdot i_1 < \cdots < i_n \land i_1 \in U_1 \land \cdots \land i_n \in U_n \} \)
- Specialization quasi-ordering is \textbf{word embedding} \( \leq^* \)
- **Theorem** (JGL 2013). \( X \) Noetherian iff \( X^* \) Noetherian
  Generalizes Higman’s Lemma (Higman 1952): \( X \) wqo iff \( X^* \) wqo
Infinite words

- Let $X^{\leq \omega} = \{\text{finite or infinite words on } X\}$ with asymptotic word topology:
  
  subbasic open sets $\langle U_1, \ldots, U_n \rangle = X^* U_1 X^* \cdots X^* U_n X^{\leq \omega}$,
  
  and $\langle U_1, \ldots, U_n; (\infty) V \rangle = X^* U_1 X^* \cdots X^* U_n (X^* V)^\omega$ ($U_i, V$ open in $X$)

- The logical view: words as specific infinite structures,
  
  $F ::= i \in U \mid i < j \mid \bot \mid \top \mid F \lor F \mid F \land F \mid \exists i, F$

  $\mid \exists^\infty i, G$

  $G ::= i \in U \mid i < j \mid \bot \mid \top \mid G \lor G$

- Specialization quasi-ordering is (infinite) word embedding

- **Theorem** (JGL 2021). $X$ Noetherian iff $X^{\leq \omega}$ Noetherian
  
  No equivalent in wqo theory — except if you adopt bqo theory.
Transfinite words

- Let $X^{<\alpha} = \{\text{ordinal-indexed words on } X \text{ of length } < \alpha\}$
- **Regular subword topology** better described through subbasic closed sets $F_1^{<\alpha_1} \cdots F_n^{<\alpha_n}$ where each $F_i$ is closed in $X$ and each $\alpha_i$ is an ordinal
- Contains $X^* = X^{<\omega}$ and $X^{\leq\omega} = X^{<\omega+1}$ as special cases

- **Theorem** (JGL, Halfon, Lopez 2022, submitted).
  $X$ Noetherian iff $X^{<\alpha}$ Noetherian
  No equivalent in wqo theory — except if you adopt bqo theory…
  but specialization quasi-ordering is not word embedding in general.
Topological WSTS
❖ So Noetherian spaces go beyond wqos, but do they have any use?
❖ Of course they do: a reminder of where they come from
❖ An application in verification
The origin of Noetherian spaces

- The spectrum $\text{Spec}(R)$ of a ring $R$ is the set of its prime ideals $p$
- with the Zariski topology, whose closed subsets are $\{p \in \text{Spec}(R) \mid I \subseteq p\}$, where $I$ ranges over the ideals of $R$
- **Fact.** The spectrum of a Noetherian ring is Noetherian. (every monotone chain of ideals is stationary)
- In particular if $R = K[X_1, \dots, X_n]$ for some Noetherian ring, e.g., $\mathbb{Z}$
- One can compute with ideals, represented by Gröbner bases (Buchberger 1976)
An application of Gröbner bases in verification

- Verification of polynomial programs (Müller-Olm & Seidl 2002)

- Propagates ideals of $\mathbb{Z}[X_1, \ldots, X_n]$ backwards, as in the Pre* algorithm ($X_1, \ldots, X_n$ = variables of the program)

- Terminates because every monotonic chain $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$ of ideals is stationary

- ... very similar to Pre* on WSTS, but the (infinite) transition system underlying a polynomial program is not a WSTS (inclusion between ideals not a wqo)
Definition. A topological WSTS is a transition system \((X, \rightarrow)\) with a Noetherian topology \(\leq\) on \(X\) satisfying lower semicontinuity:

for every open subset \(U\), \(\text{Pre}(U)\) is open

- Namely, replace \(wqo\) by Noetherian
  monotonicity by lower semicontinuity
- If the topology is Alexandroff, then Noetherian=\(wqo\), lower semicontinuity=monotonicity
  In particular, every WSTS is a topological WSTS
- Polynomial programs are topological WSTS
  — in the Zariski topology of \(\text{Spec}(\mathbb{Z}[X_1, \ldots, X_n])\)
Topological coverability is decidable

**Topological coverability:**

**INPUT:** an initial configuration $x_0$, an open set $U$ of bad configurations

**QUESTION:** is there a $x \in U$ such that $x_0 \rightarrow^* x$?

**An effective** topological WSTS is one where:
- open sets are representable
- $\subseteq$ is decidable on open sets
- $U \mapsto \text{Pre}(U)$ is computable

**Theorem** (JGL 2011.) Topological coverability is **decidable** on effective topological WSTSs.

The algorithm is the same as with WSTSs.

```
fun pre* U =  
    let V = pre U  
    in  
        if V \subseteq U  
        then U  
        else pre* (U \cup V)  
    end; 

fun coverability (s, B) =  
    s in pre* (B); 
```
Concurrent polynomial programs

(JGL 2011)

+ Finite networks of polynomial programs $P_1, \ldots, P_m$ communicating through lossy communication queues on a finite alphabet $\Sigma$

+ State space = finite product of
  - spectra of polynomial rings $\mathbb{Z}[X_1, \ldots, X_n]$, one for each $P_i$
  - $\Sigma^*$, with word topology, one for each communication queue

This is Noetherian, because:

+ Proposition. Any finite product of Noetherian spaces is Noetherian.
Concurrent polynomial programs

(JGL 2011)

- Those are topological WSTSSs (lossiness necessary)
  Hence:

- **Theorem** (JGL 2011).
  Topological coverability is **decidable** for concurrent polynomial programs.

- You still have to prove effectivity.
  For that, you need to find a representation for open sets.
  But open sets are **no longer** of the form \( \uparrow \{ x_1, \ldots, x_n \} \)
Representations, sobrifications
Representing open sets: the trick

* Embed state space $X$ into its **sobrification** $X^s$

Oops, I have not said what that was, have I?
A closed set $F \in \mathcal{H}X$ is \textbf{irreducible} iff for all $F_1, \cdots, F_n \in \mathcal{H}X$, $F \subseteq \bigcup_i F_i \Rightarrow \exists i, F \subseteq F_i$

Every set $\downarrow x$ is irreducible closed
$X$ is \textbf{sober} iff $T_0$
and those are the only irreducible closed sets

The \textbf{sobrification} $X^s = \{ F \in \mathcal{H}X \mid F \text{ irreducible} \}$, seen as subspace of $\mathcal{H}X$ is always \textbf{sober}, and $X$ embeds into $X^s$ through $x \mapsto \downarrow x$

$X$ and $X^s$ have isomorphic lattices of open subsets
Representing open sets: the trick

- Embed state space $X$ into its **sobrification** $X^s$
- Both have isomorphic lattices of open sets
- Represent open sets $U$ by their complements: **closed sets** $C$
- Now: 
  In a sober Noetherian space, every closed set $C$ is a **finitary** subset $\downarrow \{x_1, \ldots, x_n\}$.
- Hence we can represent $U$ by 
  (the complement of the downward closure in $X^s$) 
  of **finitely many** points… in $X^s$
Representing points in sobrifications

- For a finite set $\Sigma$, with the discrete topology, $\Sigma^s = \Sigma$
- Products: $(X \times Y)^s = X^s \times Y^s$
- $\text{Spec}(\mathbb{Z}[X_1, \cdots, X_n])$: already sober, points = prime ideals, represented as Gröbner bases
- $(X^*)^s$ consists of word products $\mathbb{P} ::= \epsilon | C^?P | F^*P$
  with $C \in X^s$, $F = C_1 \cup \cdots \cup C_n$ ($C_i \in X^s$)
- All those are representable on a computer (Finkel, JGL 2009, 2021)
Statures of Noetherian spaces
Maximal order types of well-partial-orderings

Statures of Noetherian spaces as generalization of maximal order types

... we are not really changing the subject,
and we will use the representations of points in $X^s$ again
A well-partial-ordering is a well-quasi-ordering that is antisymmetric.

**Theorem** (Wolk 1967). A wpo is a partial ordering whose linear extensions are all well-founded.

*Note:* every linear well-founded ordering is isomorphic to a unique ordinal, … its order type.

**Theorem** (de Jongh, Parikh 1977). Among those, one has maximal order type.

Any meaningful equivalent of that notion for Noetherian spaces? But first, why should we bother about maximal order types anyway?
Why bother about maximal order types?

❖ First studied by de Jongh and Parikh (1977) then Schmidt (1979)


❖ Ordinal complexity of the size-change principle for proving the termination of programs and rewrite systems Blass and Gurevich (2008)

❖ and…
Why bother about maximal order types?

- Figueira, Figueira, Schmitz and Schnoebelen (2011), Schmitz and Schnoebelen (2011) (and others) obtain **complexity upper bounds** for algorithms whose termination is based upon wqo arguments (e.g., coverability)
  - E.g., coverability in lossy channel systems is $F_{\omega \omega}$-**complete**. (way larger than Ackermann)

**Theorem 5.3** (Main Theorem).

Let $g$ be a smooth control function eventually bounded by a function in $\mathcal{F}_\gamma$ and let $A$ be an exponential nwqo.

Then $L_{A,g}$ is bounded by a function in:

- $\mathcal{F}_\gamma$ if $\gamma < \omega$ (e.g., if $g$ is primitive recursive) and $\beta \geq \omega$
- $\mathcal{F}_{\gamma+\beta}$ if $\gamma \geq 2$ and $\beta < \omega$.

From S. Schmitz, Ph. Schnoebelen, Multiply-recursive upper bounds with Higman’s Lemma. ICALP 2011.
Let us return to the question of finding a Noetherian analogue of maximal order types.
A wrong idea: minimal $T_0$ topologies

- Partial ordering $\sim T_0$ topology
  Extension $\sim$ coarser $T_0$ topology
  Linear extension $=$ maximal extension $\sim$ minimal $T_0$ topology

  A minimal $T_0$ topology is necessarily the upper topology of a linear ordering.

- Unfortunately, minimal $T_0$ topologies do not exist in general:
  \textbf{Fact.} $\mathbb{R}_{\text{cof}}$ is Noetherian, but has no coarser minimal $T_0$ topology.
  (Its uncountably many proper closed subsets would all have to be finite, and linearly ordered.)
Theorem (Kříž 1997, Blass and Gurevich 2008).
Maximal order type of a wpo \((X, \leq)\)

= **ordinal rank** \(\|X\|\) of the top element \(X\)
in the poset \((\mathcal{D}X, \subseteq)\) of downwards-closed subsets of \(X\).

Ordinal rank inductively defined by:

\[\|F\| = \sup\{\|F'\| + 1 \mid F' \in \mathcal{D}X, F' \subsetneq F\}\]

Example: \(X = \{0,1,2\}\), ordered by equality

maximal order type=3

The stature of \(X\)
Statures of Noetherian spaces

Definition. The stature of a Noetherian space $X$ is the ordinal rank $||X||$ of the top element $X$ in the poset $(\mathcal{H}X, \subseteq)$ of closed subsets of $X$.

||$F|| = \sup\{||F'|| + 1 \mid F' \in \mathcal{H}X, F' \subset F\}$

Matches previous definition: for a wqo in its Alexandroff topology, closed = downwards-closed $\mathcal{H}X = \mathcal{D}X$

$X$ is Noetherian iff:
(6) Every antitonic chain $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ of closed subsets is stationary
(7) $\mathcal{H}X$ is well-founded.
The stature of a finite space

❖ For a finite $T_0$ space $X$, $||X|| = \text{card } X$

... a finite $T_0$ space is just a finite poset, so that was known already
The stature of $X^*$

- **Theorem** (JGL, Laboureix 2022). If $X \neq \emptyset$ is Noetherian and $\alpha = ||X||$, then $||X^*|| = \omega^{\omega^{\alpha \pm 1}}$

  (+1 if $\alpha = \epsilon_\rho + n$, –1 if $\alpha$ finite)

- Not very surprising: already known when $X$ wqo (Schmidt 1979)

- The proof is very different, and is **localic**.
  Explicitly, we do not reason on points (words),
  but on **closed sets** = finite unions of word products

$(X^*)^c$ consists of **word products**

$P ::= \epsilon \mid C^2P \mid F*P$

with $C \in X^c$, $F = C_1 \cup \cdots \cup C_n$ ($C_i \in X^c$)
An excerpt from the proof of $||X^*|| \geq \omega^{\omega^{\alpha+1}}$

- For a well-founded poset $P$, let $\text{Step}(P) = \{(p, q) \mid p < q\}$, strictly ordered by $(p, q) < (p', q')$ iff $q \leq p'$

**Lemma.** The ordinal height of $\text{Step}(P)$ is that of $P$ minus 1.

(convention: for a non-successor ordinal $\beta$, $\beta - 1 = \beta$)

- Given $F \in HX$ and $C \in X^s$ such that $C \not\subseteq F$, $(F, F \cup C) \in \text{Step}(HX)$

- Let $C_0 = \emptyset$, $C_{n+1} = (F^*C^?)^n F^*$, $\mathcal{A}_n = \{A \in HX \mid C_n \subseteq A \subsetneq C_{n+1}\}$

- Map $(B, B^+) \in \text{Step}(H(F^*))$, $A \in \mathcal{A}_n$ to $(F^*C^?)^{n+1}B \cup AC^?B^+ \cup C_{n+1}$

- This is strictly monotonic: $\text{Step}(H(F^*)) \times_{\text{lex}} \mathcal{A}_n \rightarrow \mathcal{A}_{n+1}$

A finite union of word products
An excerpt from the proof of $||X^*|| \geq \omega^{\omega^{\alpha \pm 1}}$

- There is a **strictly monotonic** map: $\text{Step}(\mathcal{H}(F^*)) \times_{\text{lex}} \mathcal{A}_n \to \mathcal{A}_{n+1}$

- The ordinal height of $\mathcal{A}_{n+1}$ is $||C_{n+1}||$

  Hence if $||F^*|| \geq \omega^\omega$ then $||C_{n+1}|| \geq \omega^{\omega \times (n+1)}$,

  so $||(F \cup C)^*|| \geq \omega^{\omega^{\beta+1}}$, by taking suprema over $n \in \mathbb{N}$

- This is the key step in a well-founded induction on $F \in \mathcal{H} X$

  showing $||F^*|| \geq \omega^{\omega^{||F|| \pm 1}}$

- Finally, let $F = X$; by definition, $||X|| = \alpha$. $\square$
The stature of $\mathbb{Z}[X_1, \ldots, X_n]$

- The ordinal height of the lattice of ideals of $\mathbb{Z}[X_1, \ldots, X_n]$ is $\omega^n + 1$ (Aschenbrenner, Pong 2004)

- Hence $||\text{Spec}(\mathbb{Z}[X_1, \ldots, X_n])|| \leq \omega^n$ (and I conjecture equality—not checked out of laziness)

- Only indirectly related to wqos (through leading monomials) but remember that $\text{Spec}(\mathbb{Z}[X_1, \ldots, X_n])$ is not itself wqo.

- Together with $||X \times Y|| = ||X|| \otimes ||Y||$ (JGL, Laboureix 2022) extending the same formula on wqos (de Jongh, Parikh 1977), we obtain the stature of the state space of concurrent polynomial programs...
The stature of the state space of concurrent polynomial programs

- \( m \) programs, each on \( n \) variables
- \( p \) queues, on \( k \geq 1 \) letters

- Stature of state space \( \leq (\omega^n)^m \otimes (\omega^{\omega^{k-1}})^p \)
  \[ = \omega^{nm} \oplus \omega^{k-1} \cdot p \]

- Note that the contribution of the polynomial programs (\( nm \)) is much lower than the contribution of the queues (\( \omega^{k-1} \cdot p \))

- What is the actual complexity of verifying concurrent polynomial programs?
Our findings on statures so far

- We have already obtained **statures** of quite a few Noetherian constructions
- We retrieve the known formulae from wqo theory, which **extend** properly
- and **new formulae** for non-wqo Noetherian spaces
- A related notion: **sobrification ranks** $|X^s|$ 
- Missing: finite **trees**, notably (see Schmidt 1979 for the wqo case)
- Application to actual **complexity** upper bounds?

| $X$      | $|X|$ | $\text{sob }X$ |
|----------|------|---------------|
| finite $T_0$       | $\text{card }X$ | $\leq \text{card }X$ |
| ordinal $\alpha$ (Alex.) | $\alpha$ | $\alpha / \alpha+1$ |
| $Y+Z$        | $\max(\text{sob }Y, \text{sob }Z)$ |
| $Y^+_{\leq \alpha}Z$ | $\text{sob }Y+\text{sob }Z$ |
| $1+ Y^+$ | $1+\text{sob }Y$ |
| $\omega^{|Y|+1}$ | $(\text{sob }Y+\text{sob }Z)-1$ |
| $\omega^\|Y\|+1$ | $\omega^{|Y|+1}$ |
| $\alpha / \alpha-1$ | $\alpha / \alpha+1$ |
| $\min(\text{card }Y, \omega)$ | $1 / 2$ |
| $1+|Y|\ldots\omega^{|Y|+1}$ | $|Y|+1$ |
| $\omega^{|\omega^{|Y|+1}|}$ | $\omega^{|\omega^{|Y|+1}|+1}$ |
| $\omega^{|\omega^{|Y|+1}|+1}$ | $\omega^{|\omega^{|Y|+1}|+1}$ |
| $\leq \omega^{|\omega^{|Y|+1}|+1}$ | $\leq \omega^{|\omega^{|Y|+1}|+1}$ |


Conclusion
Conclusion, research directions

- A rich theory extending \textit{wqos} into the topological: \textbf{Noetherian} spaces
- Old results extend, new results pop up (powersets, spectra, infinite words)
- Ordinal analysis: the \textbf{stature} $||X||$ = ordinal rank of top element of $\mathcal{H}X$ as an analogue of maximal order types
- Still in its infancy