



Laboratoire  
Méthodes  
Formelles

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# Noetherian spaces, wqos, and their statures

BRAVAS

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université  
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# Outline

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- ❖ Wqos, WSTS... your five minutes of comfort
- ❖ Extending classical theorems from wqos to Noetherian spaces
- ❖ Sobrifications of Noetherian spaces, and their representations
- ❖ Statures of Noetherian spaces and maximal order types of wqos



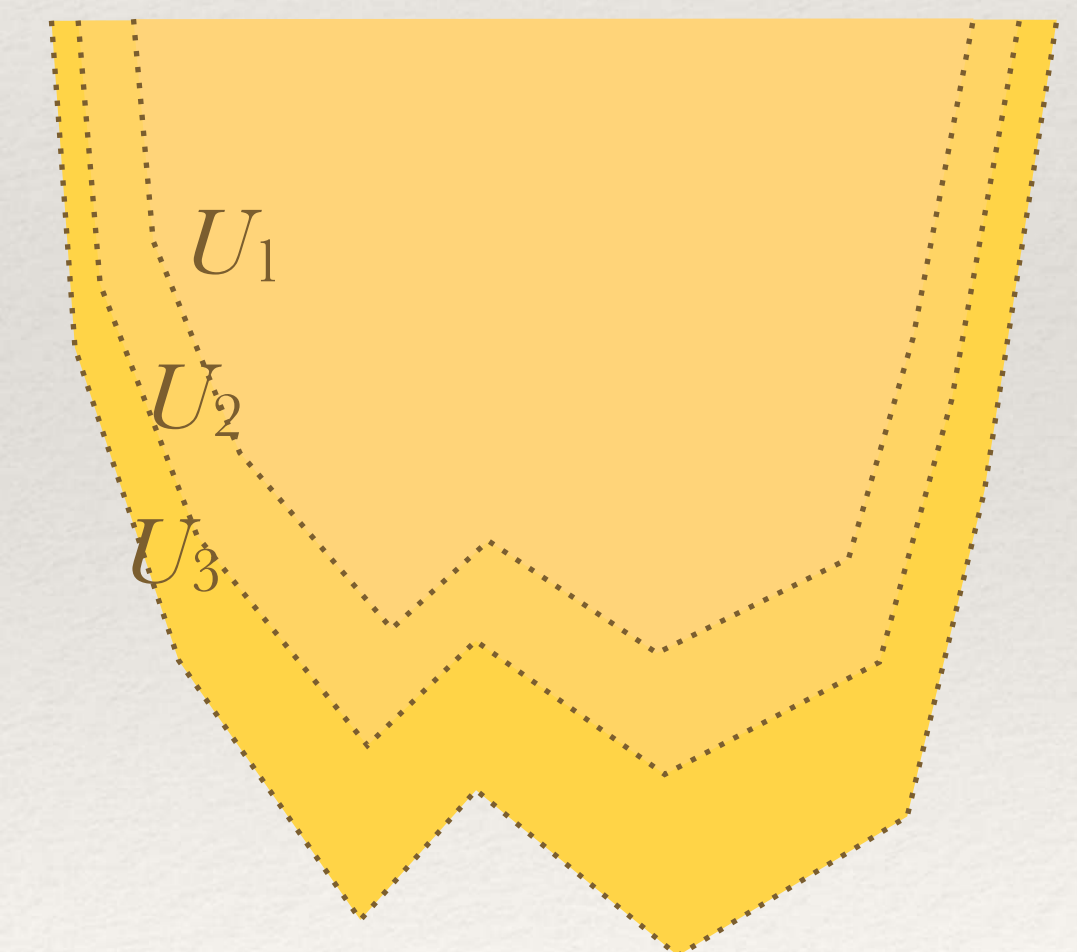
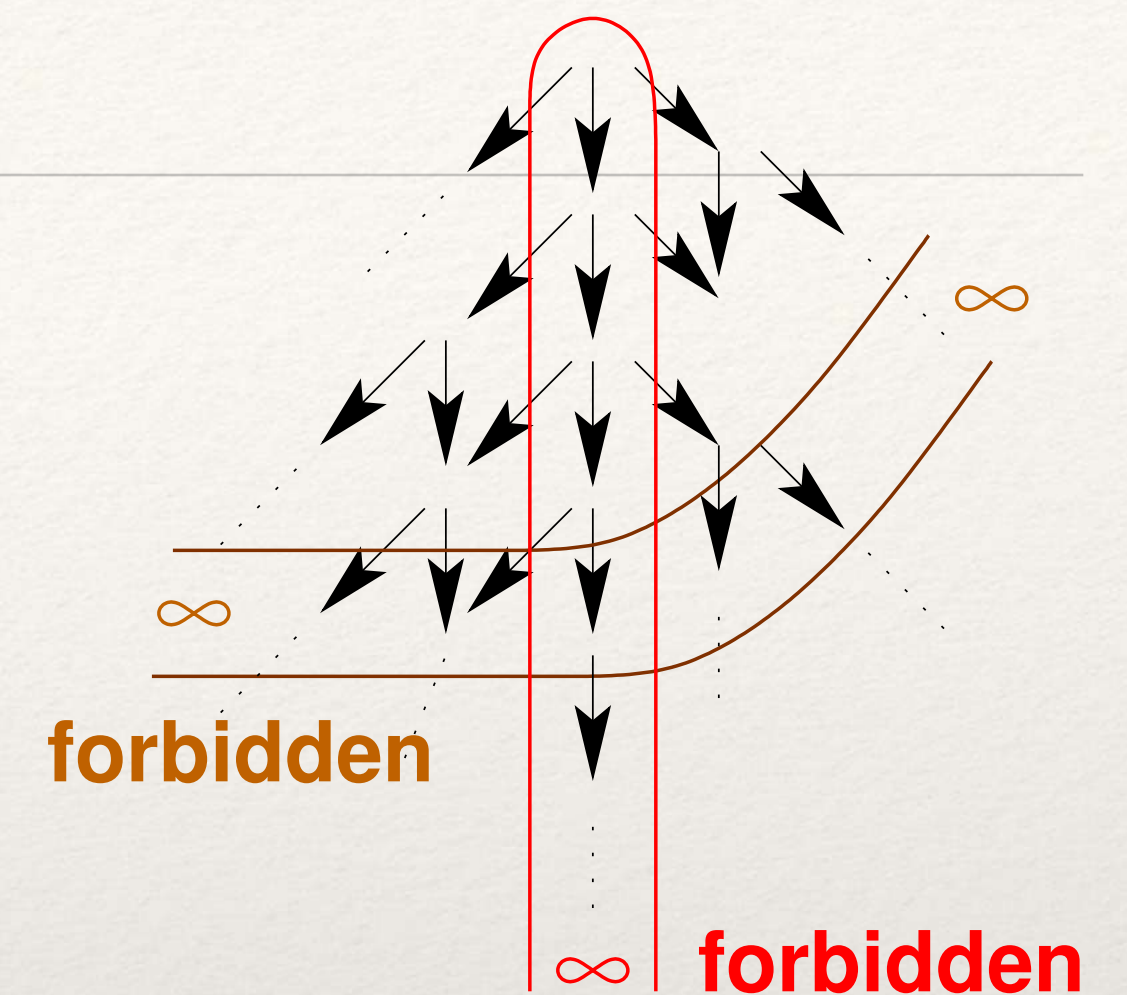
Wqos and WSTS



# Well-quasi-orders

- ❖ **Fact.** The following are equivalent for a quasi-ordering  $\leq$ :
  - (1) Every sequence  $(x_n)_{n \in \mathbb{N}}$  is **good**:  $x_m \leq x_n$  for some  $m < n$
  - (2) Every sequence  $(x_n)_{n \in \mathbb{N}}$  is **perfect**: has a monotone subsequence
  - (3)  $\leq$  is **well-founded** and has **no infinite antichain**.
  - (4) Every **upwards-closed** subset is the upwards-closure  $\uparrow \{x_1, \dots, x_n\}$  of a **finite set**
  - (5) Every monotonic chain  $U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots$  of upwards-closed subsets is **stationary**  
(i.e., all the sets  $U_n$  are equal from some rank on)

❖ **Defn.** Such a quasi-ordering  $\leq$  is called a **well-quasi-order (wqo)**.





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# Examples

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- ❖  $\mathbb{N}$ , with its usual ordering — More generally, any **total well-founded** order
- ❖ Every **finite** set, with any quasi-ordering
- ❖ Finite disjoint **sums**, finite **products** of wqos are wqo
- ❖ **Images** of wqos by monotonic maps are wqo (in particular quotients)
- ❖ **Inverse images** of wqos by order-reflecting maps are wqo (in particular subsets)
- ❖ **Higman's Lemma.** Let  $X^* = \{\text{finite words over alphabet } X\}$   
ordered by word embedding  $\leq_*$ . Then  $X \text{ wqo} \Leftrightarrow X^* \text{ wqo}$
- ❖ **Kruskal's Theorem.** Let  $\mathcal{T}(X) = \{\text{finite trees with } X\text{-labeled vertices}\}$   
ordered by homeomorphic tree embedding  $\leq_T$ . Then  $X \text{ wqo} \Leftrightarrow \mathcal{T}(X) \text{ wqo}$ .
- ❖ And so on.

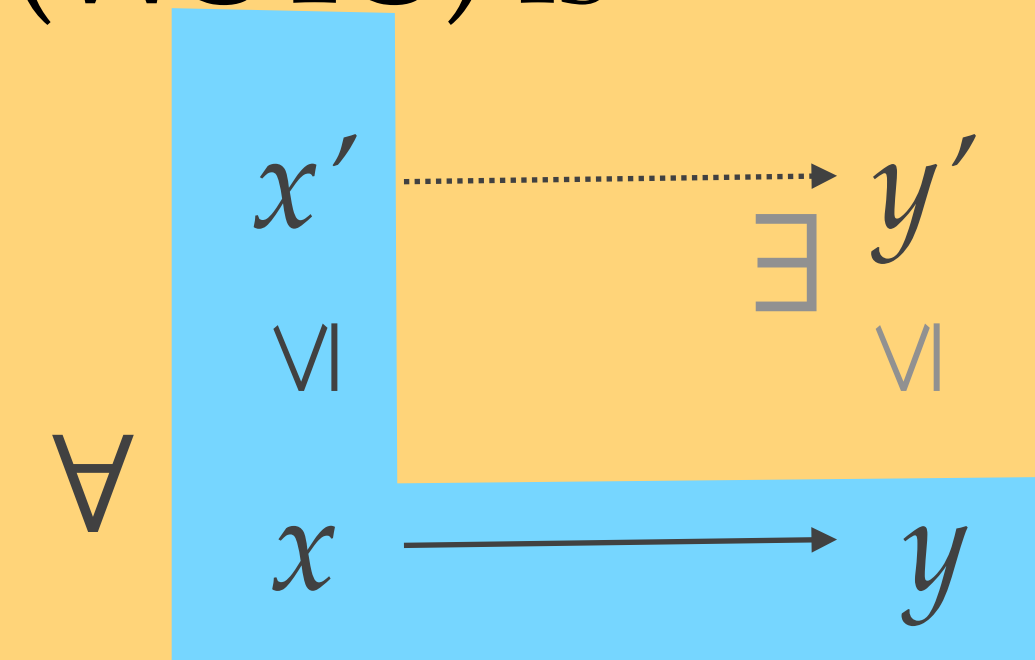


# Well-structured transition systems

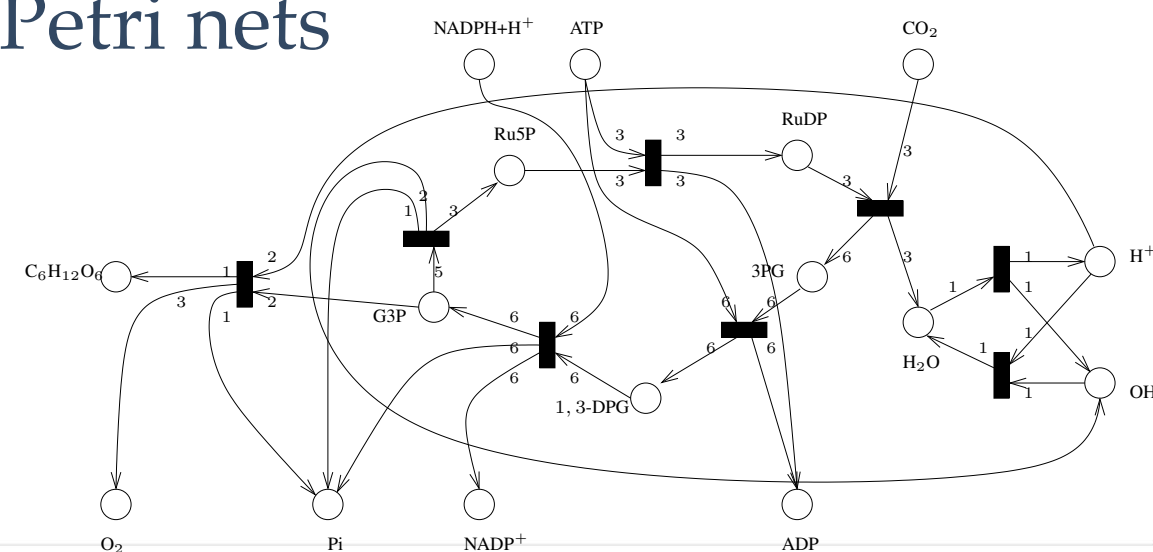
(Abdulla, Čerāns, Jonsson & Tsay 2000,  
Finkel & Schnoebelen 2001)

- ❖ A very interesting class of (infinite) transition systems where **coverability** (a special form of reachability) is **decidable**

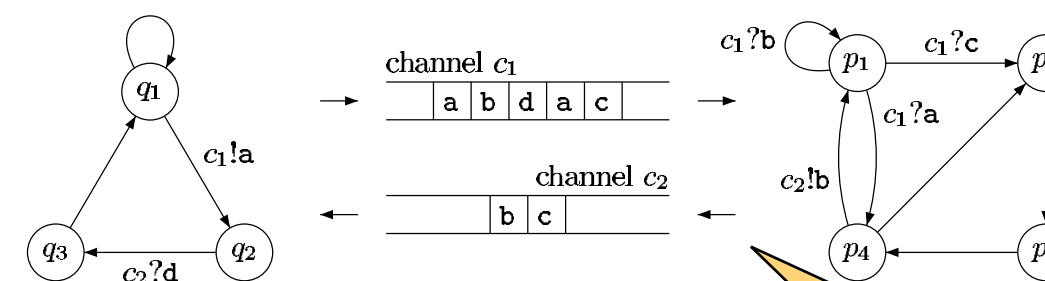
- ❖ **Definition.** A well-structured transition system (WSTS) is a transition system  $(X, \rightarrow)$  with a **wqo**  $\leq$  on  $X$  satisfying (strong) **monotonicity**:



Petri nets



Lossy channel systems



... and many other examples

letters can spontaneously vanish from communication queues  
(needed for decidability... and rather realistic)



# Coverability is decidable

- ❖ **Effective WSTSs:**

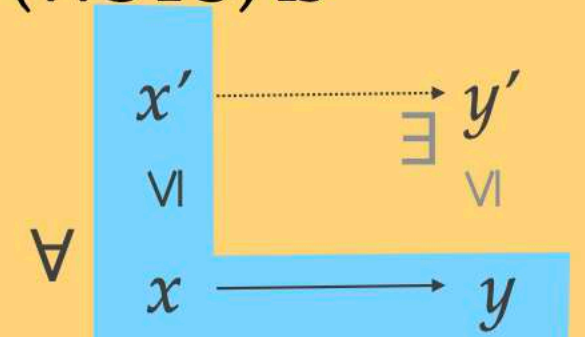
- points are representable

- $\leq$  is decidable

- $y \mapsto \{x_1, \dots, x_n\} = \text{Pre}(\uparrow y)$  is computable (so one can compute  $\text{Pre}(U)$ )

- ❖ **Theorem.** (Abdulla et al. 2000, Finkel&Schnoebelen 2001.)  
Coverability is **decidable** on effective WSTSs.

❖ **Definition.** A well-structured transition system (WSTS) is a transition system  $(X, \rightarrow)$  with a wqo  $\leq$  on  $X$  satisfying (strong) monotonicity:



```
fun pre* U =  
  let V = pre U  
  in  
    if V ⊆ U  
    then U  
    else pre* (U ∪ V)  
  end;  
  
fun coverability (s, B) =  
  s in pre* (B);
```



Beyond wqos: Noetherian spaces



# Going topological

- ❖ Every quasi-ordered set  $(X, \leq)$  gives rise to a **topological space**, whose open sets are the **upwards-closed** sets (the **Alexandroff topology**)

$X$  is wqo iff:

(5) Every monotonic chain  $U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots$  of upwards-closed subsets is **stationary**

- ❖ **Definition.** A topological space is **Noetherian** iff every monotonic chain  $U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots$  of open subsets is **stationary**.

- ❖ **Proposition.**  $(X, \leq)$  is wqo iff  $X$  is **Noetherian** in its Alexandroff topology.

- ❖ Hence Noetherian spaces generalize wqos



# Is the generalization proper?

❖ **Yes.** Consider  $\mathbb{N}_{\text{cof}}$ , the set of natural numbers with the **cofinite topology**, whose closed sets are the finite subsets (plus  $\mathbb{N}$ )

❖ It may be easier to see that  $\mathbb{N}_{\text{cof}}$  is Noetherian by realizing that:

$X$  is Noetherian iff:

(5) Every monotonic chain  $U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots$  of open subsets is **stationary**

❖ **Proposition.** A space  $X$  is Noetherian iff every **antitonic** chain  $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$  of **closed** subsets is **stationary**.

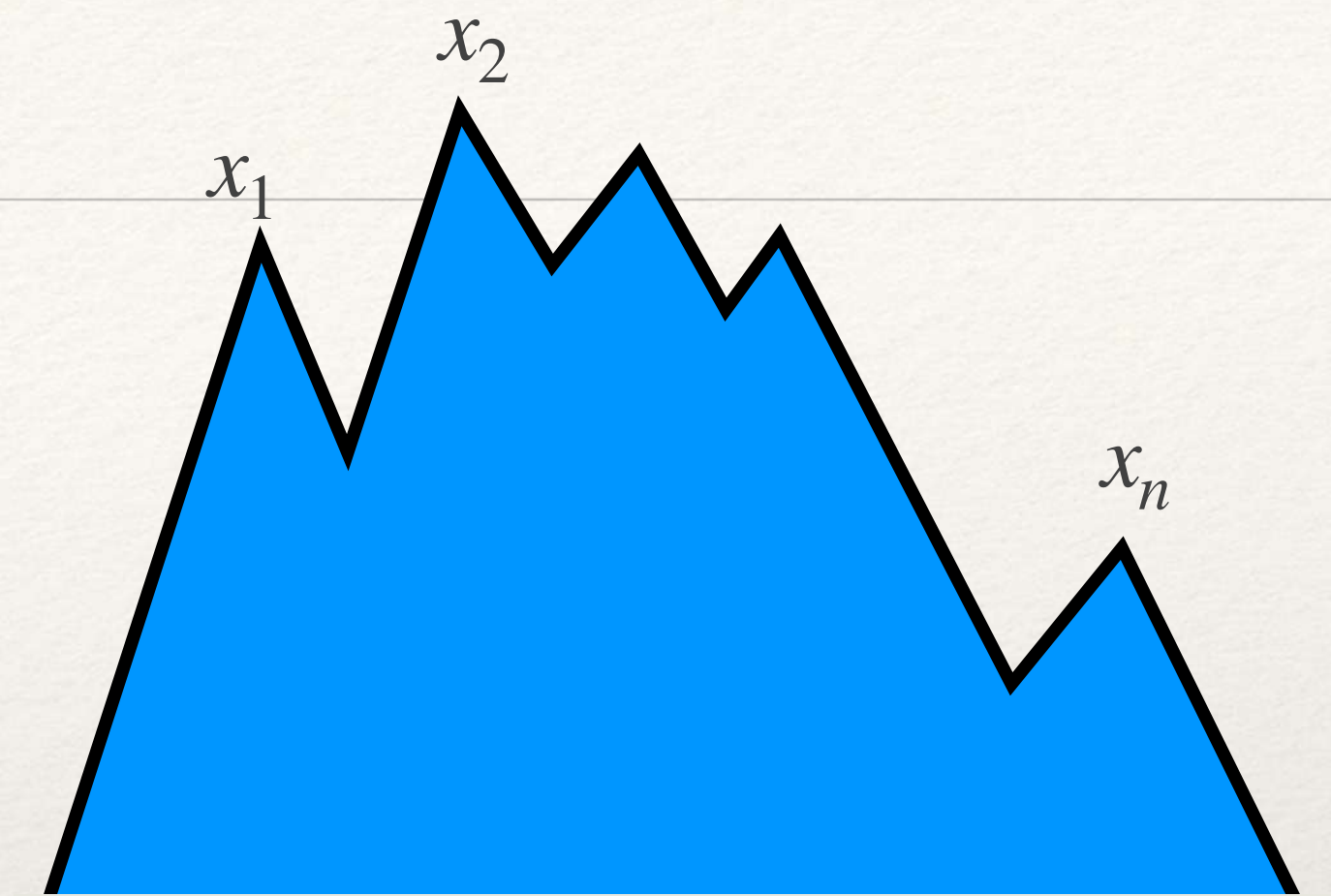
(Take complements.)

❖  $\mathbb{N}_{\text{cof}}$  does **not** arise from a wqo, because its specialization ordering is  $=$ , which is never wqo on an infinite set



# Properties T and W

- ❖ Let  $(X, \leq)$  be a quasi-ordered set.  
Its **finitary** subsets are  $\downarrow \{x_1, \dots, x_n\}$
- ❖ The finitary subsets generate the **upper topology**  
It, too, has  $\leq$  as specialization quasi-ordering



- ❖ **Proposition.** If:
  - $X$  is **well-founded**
  - (Property T)  $X$  is **finitary**
  - (Property W) For all  $x, y \in X$ ,  $\downarrow x \cap \downarrow y$  is **finitary**then  $X$  is **Noetherian** in the upper topology  
and the **closed** sets are the **finitary subsets**.

The upper topology is the **coarsest** topology  
with  $\leq$  as specialization  
The Alexandroff topology is the **finest**.

This turns out to be the general form  
of all **sober** Noetherian spaces.



# Well-founded trees

❖ Every well-founded tree  
(even not finitely branching)  
is **Noetherian** in the upper topology

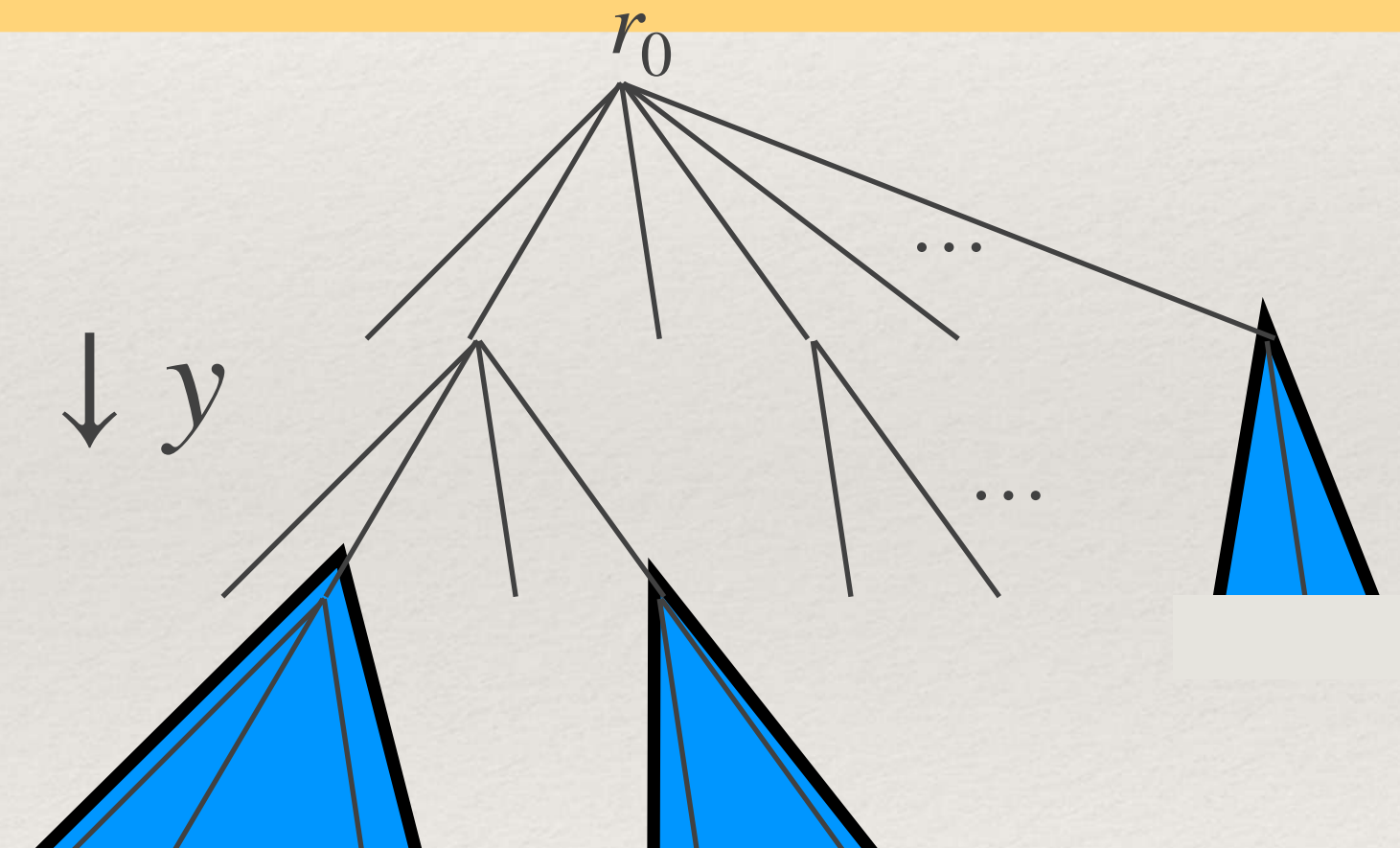
❖ **Proposition.** If:  
—  $X$  is **well-founded**  
— (Property T)  $X$  is **finitary**  
— (Property W) For all  $x, y \in X$ ,  $\downarrow x \cap \downarrow y$  is **finitary**  
then  $X$  is **Noetherian** in the upper topology

❖ (Property T)  $X = \downarrow \{r_0\}$

❖ (Property W)  $\downarrow x \cap \downarrow y$  is empty or equal to  $\downarrow x$  or  $\downarrow y$

❖ **Not wqo** unless tree is finitely branching  
(infinite antichains)

❖ Closed sets = finite disjoint unions of **subtrees**





# The Hoare hyperspace of a Noetherian space

❖ Every well-founded inf-semilattice is Noetherian in the upper topology

❖ Let  $\mathcal{H}X = \{\text{closed subsets of } X\}$   
with the upper topology of  $\subseteq$   
(Hoare hyperspace of  $X$ )

❖  $\mathcal{H}(X)$  is an inf-semilattice, hence:

**Proposition.** If  $X$  is Noetherian, then  $\mathcal{H}X$  is Noetherian.

(That is actually an equivalence.)

❖ **Proposition.** If:

—  $X$  is well-founded

— (Property T)  $X$  is **finitary**

— (Property W) For all  $x, y \in X$ ,  $\downarrow x \cap \downarrow y$  is **finitary**

then  $X$  is **Noetherian** in the upper topology

$X$  is Noetherian iff:

(5) Every monotonic chain  $U_1 \subseteq U_2 \subseteq \dots \subseteq U_n \subseteq \dots$   
of open subsets is **stationary**

(6) Every antitonic chain  $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$   
of closed subsets is **stationary**

(7)  $\mathcal{H}X$  is well-founded.



# Finite words

- Let  $X^* = \{\text{finite words on } X\}$  with **word topology**:  
 basic open sets  $\langle U_1, \dots, U_n \rangle = X^*U_1X^*\dots X^*U_nX^*$ , where each  $U_i$  is open in  $X$

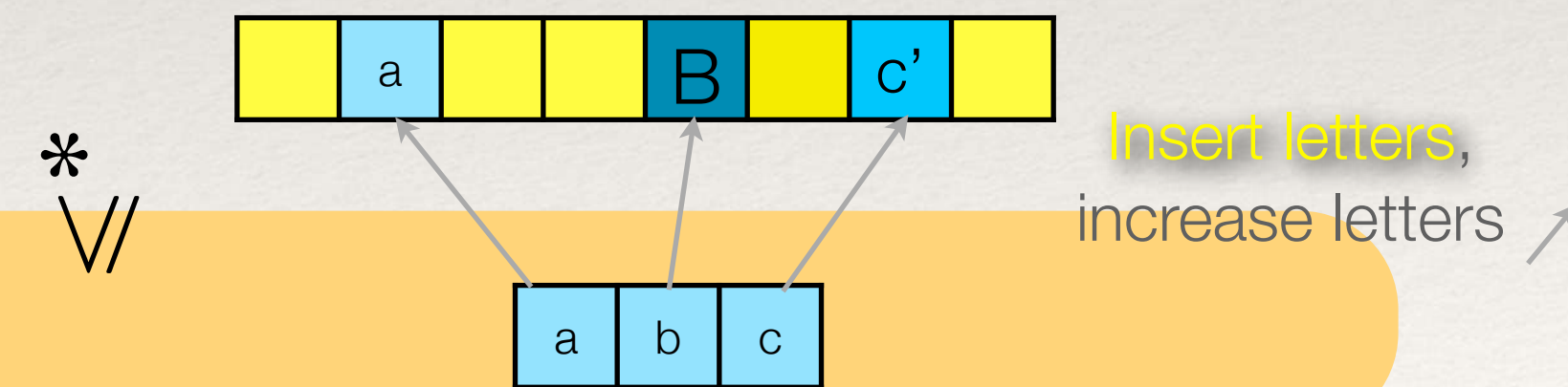
- The logical view: words as specific **finite structures**,  
 word topology = **disjunctive queries** (in database parlance)

$$F ::= i \in U \mid i < j \mid \perp \mid \top \mid F \vee F \mid F \wedge F \mid \exists i, F \quad (U \text{ open in } X)$$

letter at position  $i$  exists and is in open set  $U$

- $\langle U_1, \dots, U_n \rangle = \{\text{finite words satisfying } \exists i_1, \dots, i_n \cdot i_1 < \dots < i_n \wedge i_1 \in U_1 \wedge \dots \wedge i_n \in U_n\}$

- Specialization quasi-ordering is **word embedding**  $\leq^*$



- Theorem (JGL 2013).**  $X$  Noetherian iff  $X^*$  Noetherian

Generalizes Higman's Lemma (Higman 1952):  $X$  wqo iff  $X^*$  wqo



# Infinite words

- Let  $X^{\leq\omega} = \{\text{finite or infinite words on } X\}$  with **asymptotic word topology**:  
 subbasic open sets  $\langle U_1, \dots, U_n \rangle = X^*U_1X^*\dots X^*U_nX^{\leq\omega}$ ,  
 and  $\langle U_1, \dots, U_n; (\infty)V \rangle = X^*U_1X^*\dots X^*U_n(X^*V)^\omega$  ( $U_i, V$  open in  $X$ )

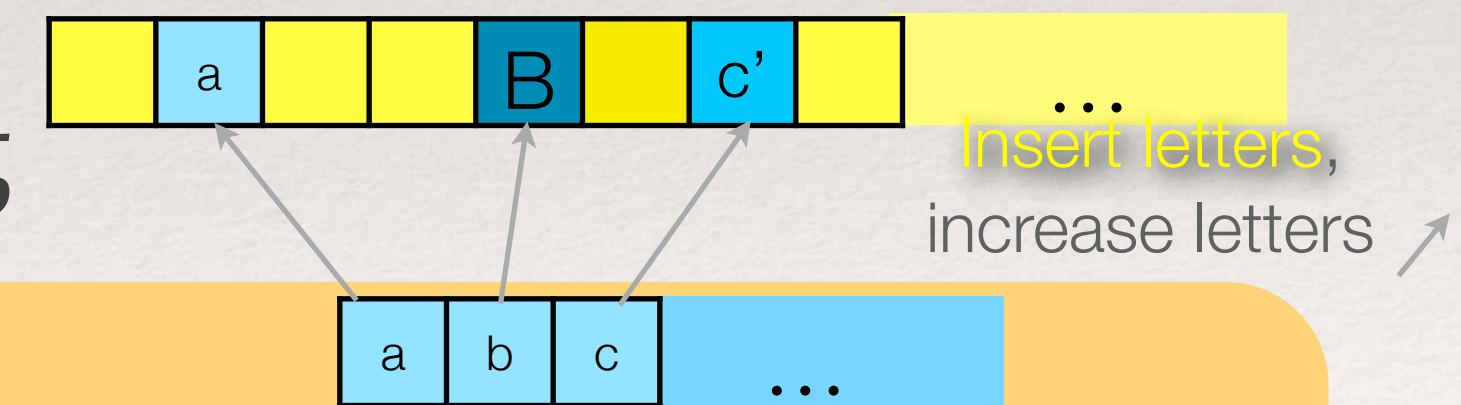
- The logical view: words as specific **infinite structures**,

$$F ::= i \in U \mid i < j \mid \perp \mid \top \mid F \vee F \mid F \wedge F \mid \exists i, F$$

$$\mid \exists^\infty i, G$$

$$G ::= i \in U \mid i < j \mid \perp \mid \top \mid G \vee G$$

- Specialization quasi-ordering is (infinite) **word embedding**



- Theorem (JGL 2021).**  $X$  Noetherian iff  $X^{\leq\omega}$  Noetherian

No equivalent in wqo theory — except if you adopt bqo theory.



# Transfinite words

- ❖ Let  $X^{<\alpha} = \{\text{ordinal-indexed words on } X \text{ of length } < \alpha\}$
- ❖ **Regular subword topology** better described through subbasic closed sets

$$F_1^{<\alpha_1} \dots F_n^{<\alpha_n}$$

where each  $F_i$  is closed in  $X$  and each  $\alpha_i$  is an ordinal

- ❖ Contains  $X^* = X^{<\omega}$  and  $X^{\leq\omega} = X^{<\omega+1}$  as special cases

- ❖ **Theorem** (JGL, Halfon, Lopez 2022, submitted).

$X$  Noetherian iff  $X^{<\alpha}$  Noetherian

No equivalent in wqo theory — except if you adopt bqo theory...  
but specialization quasi-ordering is not word embedding in general.



# Topological WSTS



- 
- 
- ❖ So Noetherian spaces go beyond wqos,  
but do they have any use?
  - ❖ Of course they do: a reminder of where they come from
  - ❖ An application in verification



# The origin of Noetherian spaces

- ❖ The **spectrum**  $\text{Spec}(R)$  of a ring  $R$  is the set of its **prime ideals**  $p$
- ❖ with the **Zariski topology**, whose closed subsets are  $\{p \in \text{Spec}(R) \mid I \subseteq p\}$ , where  $I$  ranges over the ideals of  $R$
- ❖ **Fact.** The spectrum of a Noetherian ring is Noetherian. (every monotone chain of ideals is stationary)
- ❖ In particular if  $R = K[X_1, \dots, X_n]$  for some Noetherian ring, e.g.,  $\mathbb{Z}$
- ❖ One can **compute** with ideals, represented by **Gröbner bases** (Buchberger 1976)



# An application of Gröbner bases in verification

- ❖ Verification of **polynomial programs**  
(Müller-Olm&Seidl 2002)
- ❖ Propagates ideals of  $\mathbb{Z}[X_1, \dots, X_n]$   
**backwards**, as in the  $\text{Pre}^*$  algorithm  
( $X_1, \dots, X_n =$  variables of the program)
- ❖ Terminates because every monotonic chain  $I_0 \subseteq I_1 \subseteq \dots \subseteq I_n \subseteq \dots$   
of ideals is **stationary**
- ❖ ... very similar to  $\text{Pre}^*$  on WSTS, but  
the (infinite) transition system underlying a polynomial program  
is **not** a WSTS (inclusion between ideals **not** a wqo)

```
while (*) {  
  if (*) { x=2; y=3; }  
  else { x=3; y=2; }  
  x = x*y-6; y=0;  
  if (x2-3*x*y==0)  
    while (*) { x=x+1; y=y-1; };  
  x = x2+x*y;  
}
```



# Topological WSTS

(JGL 2011)

❖ **Definition.** A **topological WSTS** is a transition system  $(X, \rightarrow)$  with a **Noetherian topology**  $\leq$  on  $X$  satisfying **lower semicontinuity**:  
for every open subset  $U$ ,  $\text{Pre}(U)$  is open

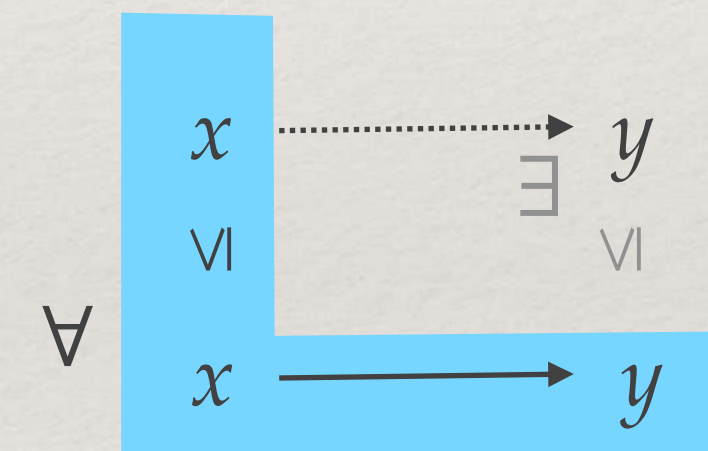
❖ Namely, replace **wqo** by **Noetherian monotonicity** by **lower semicontinuity**

❖ If the topology is Alexandroff, then **Noetherian=wqo**,  
**lower semicontinuity=monotonicity**

In particular, **every WSTS** is a topological WSTS

❖ **Polynomial programs** are topological WSTS

— in the Zariski topology of  $\text{Spec}(\mathbb{Z}[X_1, \dots, X_n])$





# Topological coverability is decidable

- ❖ **Topological coverability:**

**INPUT:** an initial configuration  $x_0$ ,  
an **open set**  $U$  of bad configurations

**QUESTION:** is there a  $x \in U$  such that  $x_0 \rightarrow^* x$ ?

- ❖ An **effective** topological WSTS is one where:

- open sets are representable
- $\subseteq$  is decidable on open sets
- $U \mapsto \text{Pre}(U)$  is computable

- ❖ **Theorem (JGL 2011.)** Topological coverability is **decidable** on effective topological WSTSs.

- ❖ The algorithm is the same as with WSTSs.

❖ **Definition.** A topological WSTS is a transition system  $(X, \rightarrow)$  with a Noetherian topology  $\leq$  on  $X$  satisfying **lower semicontinuity**:  
for every open subset  $U$ ,  $\text{Pre}(U)$  is open

```
fun pre* U =  
  let V = pre U  
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    if V ⊆ U  
    then U  
    else pre* (U ∪ V)  
end;  
  
fun coverability (s, B) =  
  s in pre* (B);
```



# Concurrent polynomial programs

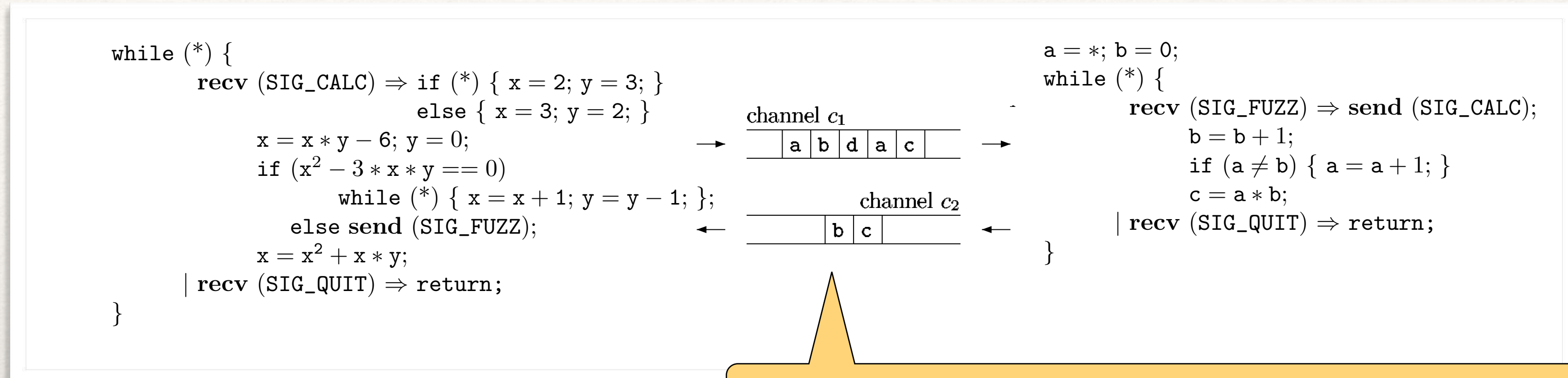
(JGL 2011)

- ❖ Finite networks of polynomial programs  $P_1, \dots, P_m$  communicating through **lossy** communication queues on a finite alphabet  $\Sigma$

- ❖ State space = finite **product** of
  - spectra of polynomial rings  $\mathbb{Z}[X_1, \dots, X_n]$ , one for each  $P_i$
  - $\Sigma^*$ , with **word topology**, one for each communication queue

This is **Noetherian**, because:

- ❖ **Proposition.** Any finite product of Noetherian spaces is Noetherian.



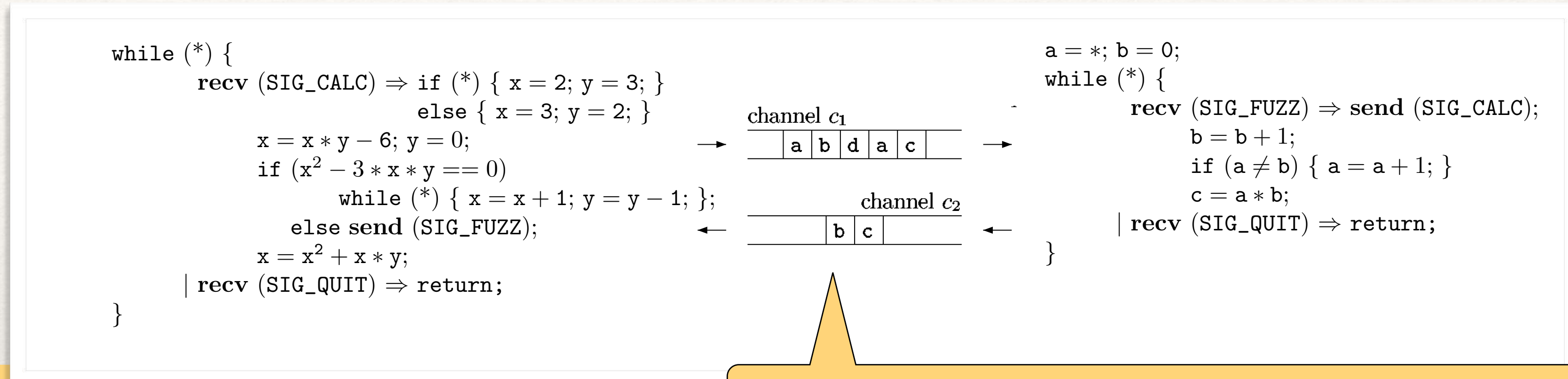
letters can spontaneously vanish from communication queues (needed for decidability... and rather realistic)



# Concurrent polynomial programs

(JGL 2011)

- ❖ Those are **topological WSTSs** (lossiness necessary)  
Hence:



- ❖ **Theorem (JGL 2011).**  
Topological coverability is **decidable** for concurrent polynomial programs.

letters can spontaneously vanish from communication queues (needed for decidability... and rather realistic)

- ❖ You still have to prove effectivity.  
For that, you need to find a representation for open sets.  
But open sets are **no longer** of the form  $\uparrow \{x_1, \dots, x_n\}$



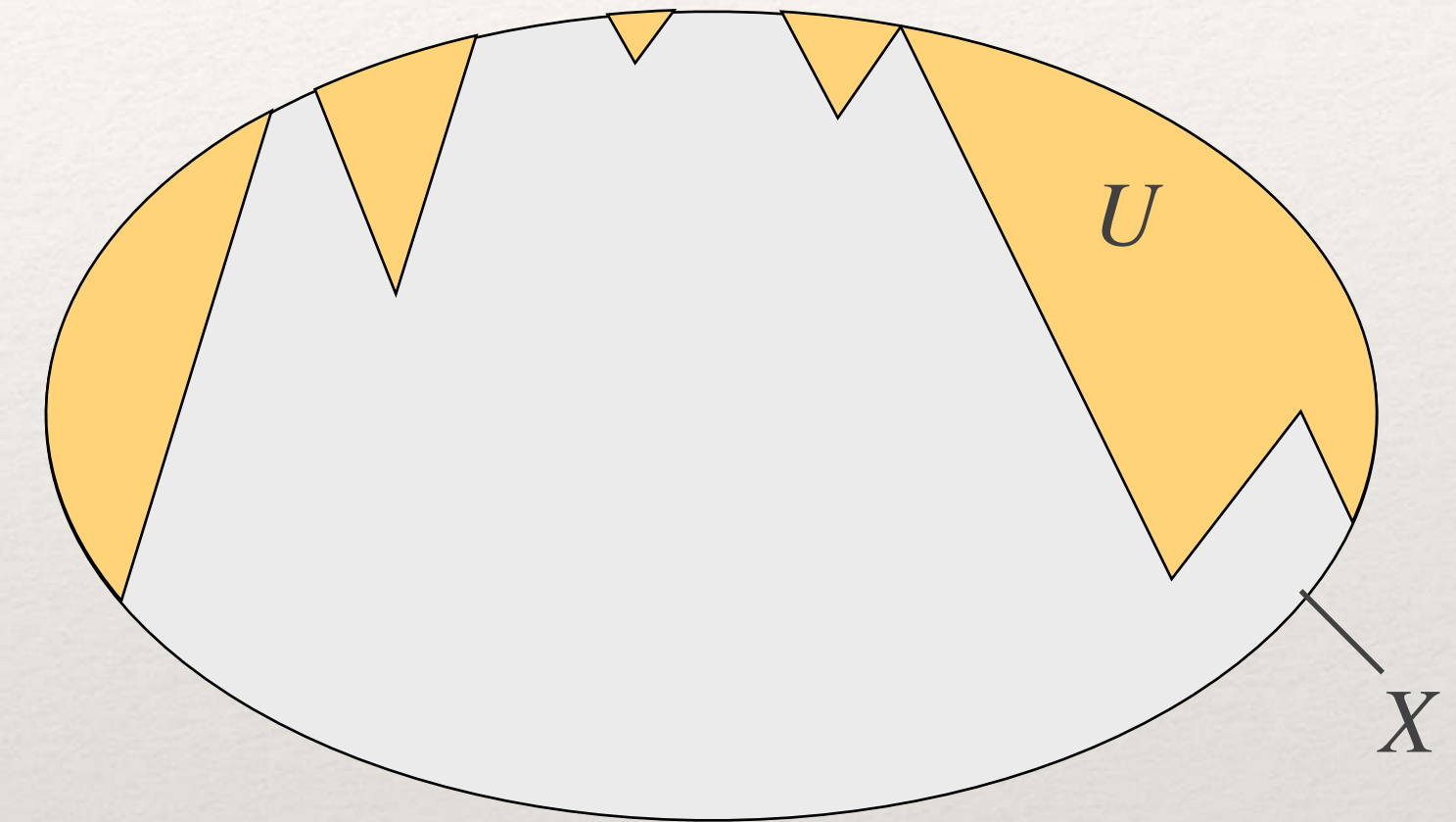
# Representations, sobifications



# Representing open sets: the trick

- ❖ Embed state space  $X$  into its **sobrification**  $X^s$

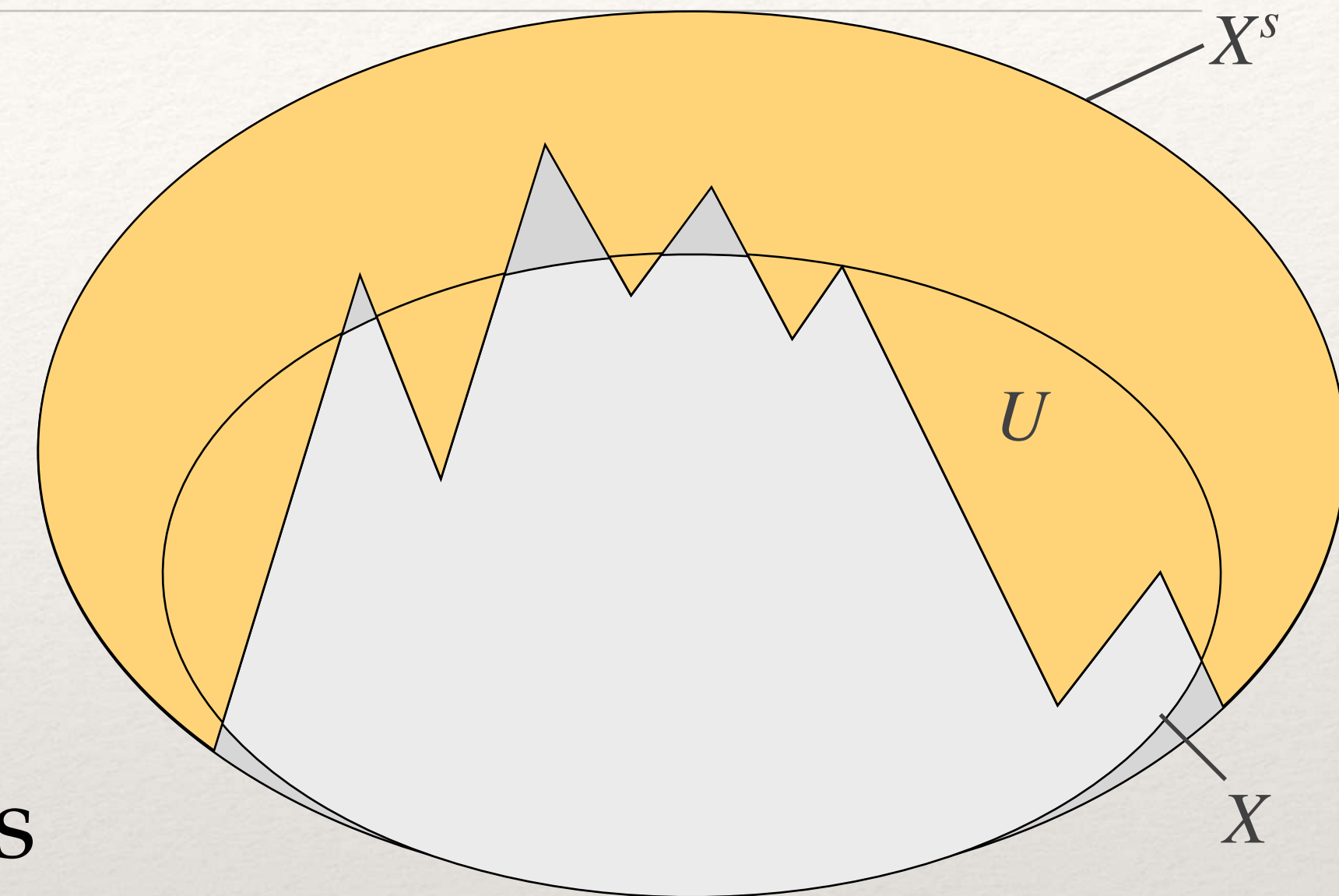
Oops, I have not said what that was,  
have I?





# Sober spaces and sobrifications

- ❖ A closed set  $F \in \mathcal{H}X$  is **irreducible** iff  
for all  $F_1, \dots, F_n \in \mathcal{H}X, F \subseteq \bigcup_i F_i \Rightarrow \exists i, F \subseteq F_i$
- ❖ Every set  $\downarrow x$  is irreducible closed  
 $X$  is **sober** iff  $T_0$   
and those are the only irreducible closed sets
- ❖ The **sobrification**  $X^s = \{F \in \mathcal{H}X \mid F \text{ irreducible}\}$ ,  
seen as subspace of  $\mathcal{H}X$  is always **sober**,  
and  $X$  embeds into  $X^s$  through  $x \mapsto \downarrow x$
- ❖  $X$  and  $X^s$  have isomorphic lattices of open subsets



E.g.,  $\mathcal{H}X, \text{Spec}(R)$ ,  
but not  $\mathbb{N}_{\text{cof}}, X^*$  for example

In particular,  
 $X$  Noetherian iff  $X^s$  Noetherian



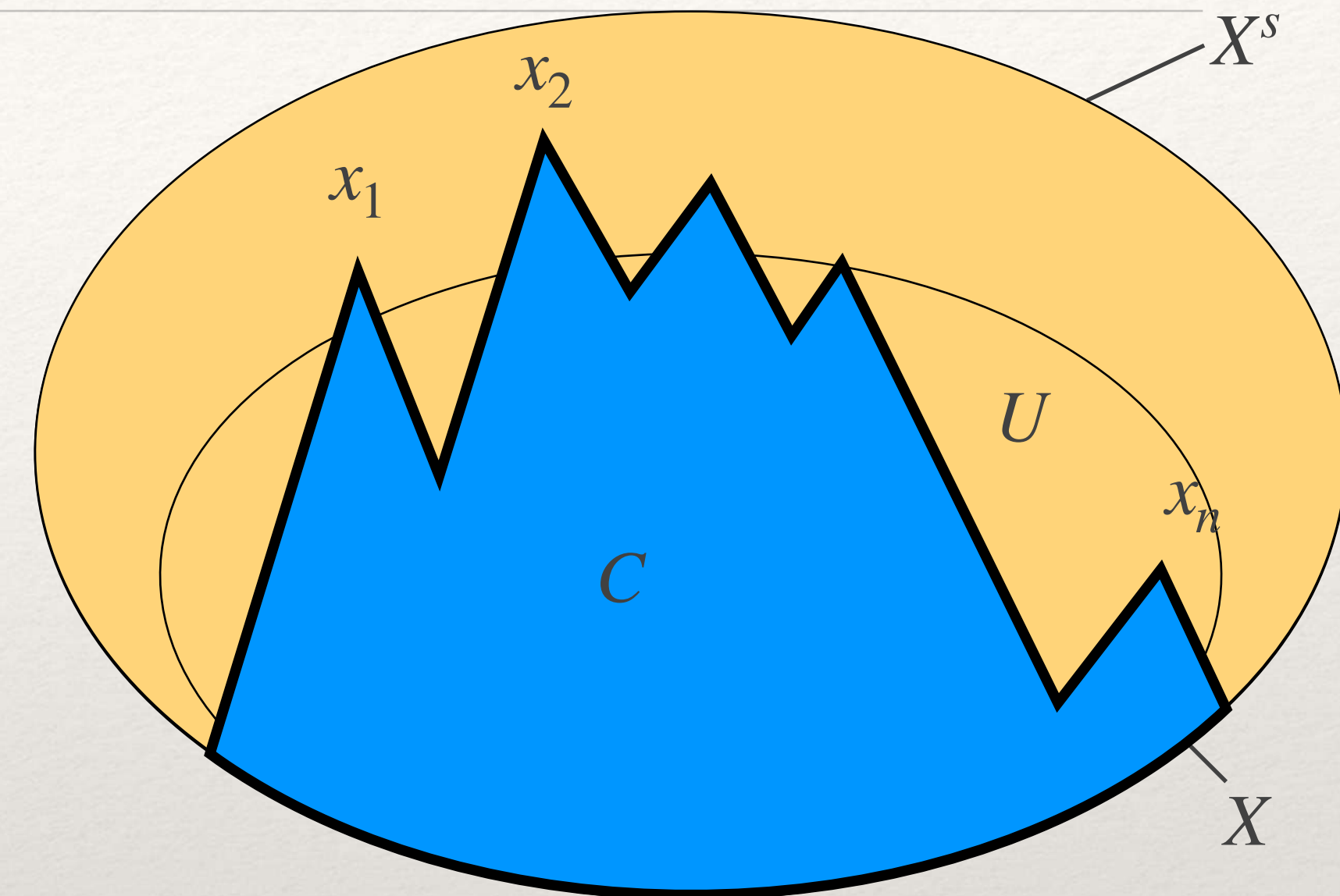
# Representing open sets: the trick

- ❖ Embed state space  $X$  into its **sobrification**  $X^s$
- ❖ Both have isomorphic lattices of open sets
- ❖ Represent open sets  $U$  by their complements: **closed sets**  $C$

❖ Now:

In a sober Noetherian space, every closed set  $C$  is a **finitary** subset  $\downarrow \{x_1, \dots, x_n\}$ .

- ❖ Hence we can represent  $U$  by  
(the complement of the downward closure in  $X^s$ )  
of **finitely many** points... in  $X^s$



## Reminder

- ❖ **Proposition.** If:
  - $X$  is well-founded
  - (Property T)  $X$  is finitary
  - (Property W) For all  $x, y \in X$ ,  $\downarrow x \cap \downarrow y$  is finitarythen  $X$  is **Noetherian** in the upper topology and the **closed** sets are exactly the **finitary** subsets.

This turns out to be the general form of all **sober** Noetherian spaces.



# Representing points in sobrifications

- For a finite set  $\Sigma$ , with the discrete topology,

$$\Sigma^s = \Sigma$$

- Products:  $(X \times Y)^s = X^s \times Y^s$

- $\text{Spec}(\mathbb{Z}[X_1, \dots, X_n])$ : already sober,

points = prime ideals, represented as Gröbner bases

- $(X^*)^s$  consists of **word products**

Embedding  $X^* \rightarrow (X^*)^s$ :  
 $abc \mapsto (\downarrow a)^?(\downarrow b)^?(\downarrow c)^?$

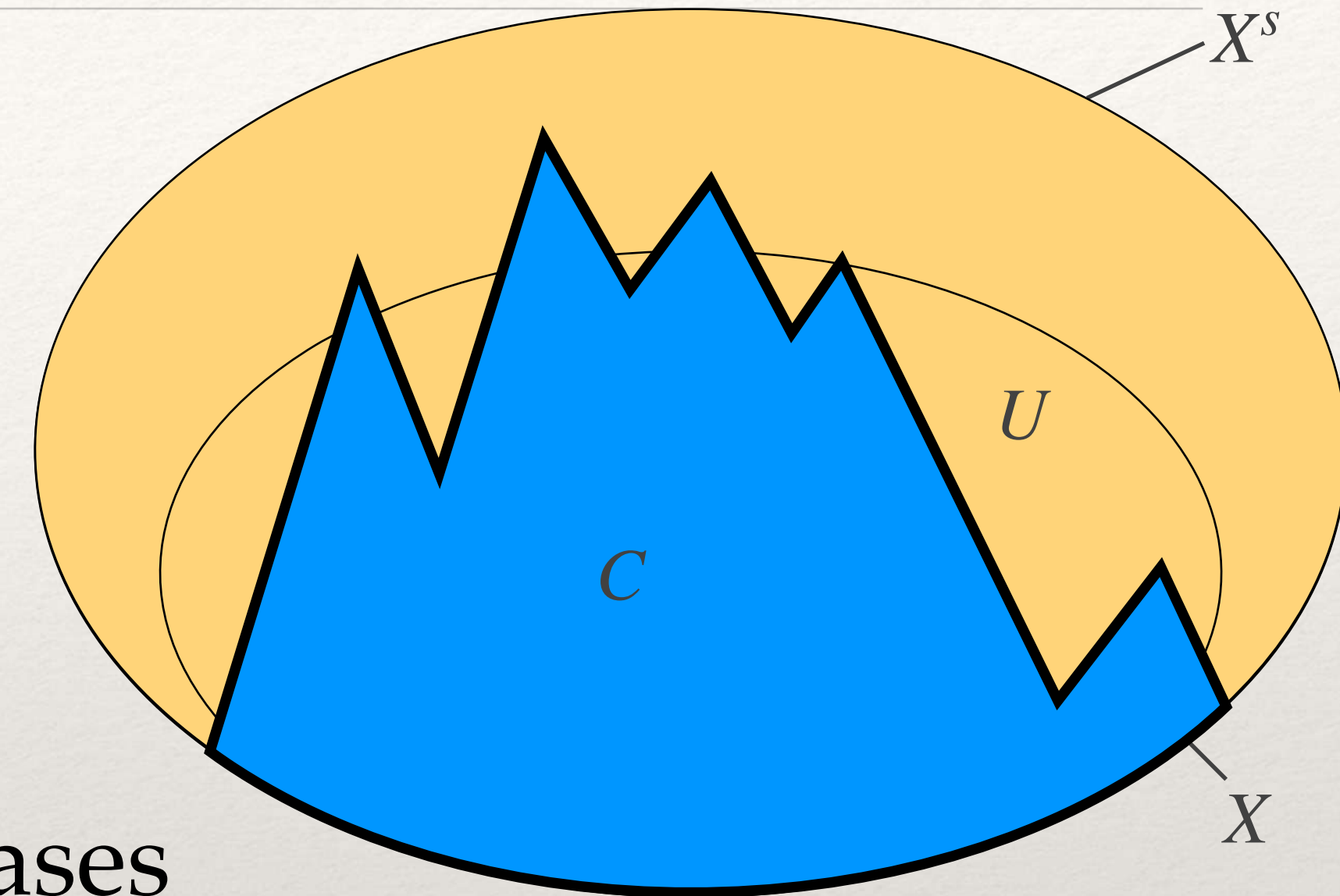
Other word products,

$$P ::= \epsilon \mid C^?P \mid F^*P$$

with  $C \in X^s, F = C_1 \cup \dots \cup C_n (C_i \in X^s)$

e.g.,  
 $C_1^?C_2^?F_1^*C_3^?F_2^*F_3^*$

- All those are **representable** on a computer (Finkel, JGL 2009, 2021)





# Structures of Noetherian spaces



- 
- 
- ❖ **Maximal order types** of well-partial-orderings
  - ❖ **Statures** of Noetherian spaces as generalization of maximal order types
  - ❖ ... we are not really changing the subject,  
and we will use the **representations** of points in  $X^S$  again



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# Maximal order types

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❖ A well-partial-ordering is a well-quasi-ordering that is antisymmetric

❖ **Theorem (Wolk 1967).** A wpo is a partial ordering whose linear extensions are all **well-founded**

**Note:** every linear well-founded ordering is isomorphic to a unique ordinal,  
... its **order type**

❖ **Theorem (de Jongh, Parikh 1977).** Among those, one has **maximal** order type.

❖ Any meaningful equivalent of that notion for Noetherian spaces?

But first, why should we bother about maximal order types anyway?



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# Why bother about maximal order types?

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- ❖ First studied by de Jongh and Parikh (1977)  
then Schmidt (1979)
- ❖ Many applications in proof theory (reverse mathematics):  
Simpson (1985), after Friedman  
van den Meeren, Rathjen, Weiermann (2014, 2015)  
etc.
- ❖ Ordinal complexity of the size-change principle for proving  
the termination of programs and rewrite systems  
Blass and Gurevich (2008)
- ❖ and...



# Why bother about maximal order types?

- ❖ Figueira, Figueira, Schmitz and Schnoebelen (2011),  
Schmitz and Schnoebelen (2011) (and others)

obtain **complexity upper bounds**  
for algorithms whose termination  
is based upon wqo arguments  
(e.g., coverability)

- ❖ E.g., coverability  
in lossy channel systems  
is  $F_{\omega^\omega}$ -complete.  
(way larger than Ackermann)

length function  
(complexity upper bound)

**Theorem 5.3 (Main Theorem).**

Let  $g$  be a smooth control function

eventually bounded by a function in  $\mathcal{F}_\gamma$

and let  $A$  be an exponential nwqo

with maximal order type  $< \omega^{\beta+1}$ .

Then  $L_{A,g}$  is bounded by a function in:

- ❖  $\mathcal{F}_\beta$  if  $\gamma < \omega$  (e.g., if  $g$  is primitive recursive) and  $\beta \geq \omega$
- ❖  $\mathcal{F}_{\gamma+\beta}$  if  $\gamma \geq 2$  and  $\beta < \omega$ .

class of functions  
elementary recursive in  $F_\beta$   
(fast-growing hierarchy)

From S. Schmitz, Ph. Schnoebelen, *Multiply-recursive upper bounds with Higman's Lemma*. ICALP 2011.



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# Going topological

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- ❖ Let us return to the question of finding a **Noetherian analogue** of maximal order types



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# A wrong idea: minimal $T_0$ topologies

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- ❖ Partial ordering  $\sim T_0$  topology  
Extension  $\sim$  coarser  $T_0$  topology  
Linear extension = maximal extension  $\sim$  minimal  $T_0$  topology
- ❖ Studied by Larson (1969).  
A minimal  $T_0$  topology is necessarily the **upper** topology of a **linear** ordering.
- ❖ Unfortunately, minimal  $T_0$  topologies do not exist in general:  
**Fact.**  $\mathbb{R}_{\text{cof}}$  is Noetherian, but has no coarser minimal  $T_0$  topology.  
(Its uncountably many proper closed subsets would all have to be finite, and linearly ordered.)



# Statures of wpos

- ❖ **Theorem** (Kříž 1997, Blass and Gurevich 2008).

Maximal order type of a wpo  $(X, \leq)$

= **ordinal rank**  $||X||$  of the top element  $X$

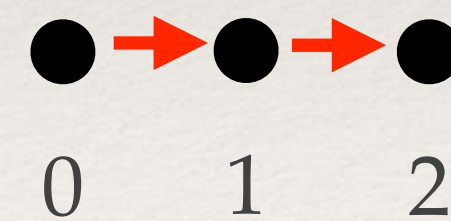
in the poset  $(\mathcal{D}X, \subseteq)$  of downwards-closed subsets of  $X$

The stature of  $X$

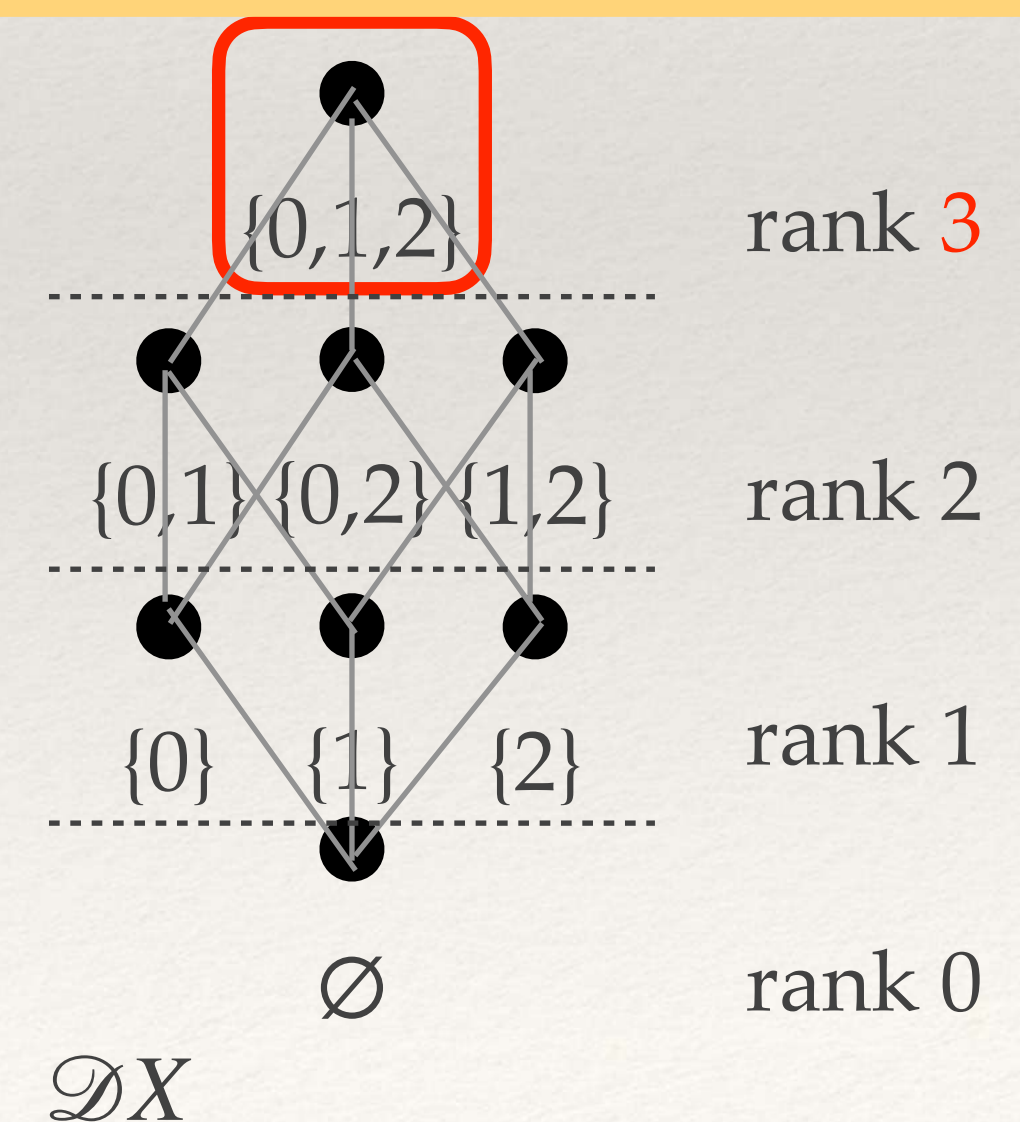
- ❖ Ordinal rank inductively defined by:

$$||F|| = \sup\{ ||F'|| + 1 \mid F' \in \mathcal{D}X, F' \subsetneq F \}$$

- ❖ **Example:**  $X = \{0,1,2\}$ ,  
ordered by equality



maximal order type=3





# Statures of Noetherian spaces

❖ **Definition.** The **stature** of a Noetherian space  $X$  is the **ordinal rank**  $||X||$  of the top element  $X$  in the poset  $(\mathcal{H}X, \subseteq)$  of **closed** subsets of  $X$

❖  $||F|| = \sup\{ ||F'|| + 1 \mid F' \in \mathcal{H}X, F' \subsetneq F \}$

❖ Matches previous definition:  
for a wqo in its Alexandroff topology,  
closed = downwards-closed

$$\mathcal{H}X = \mathcal{D}X$$

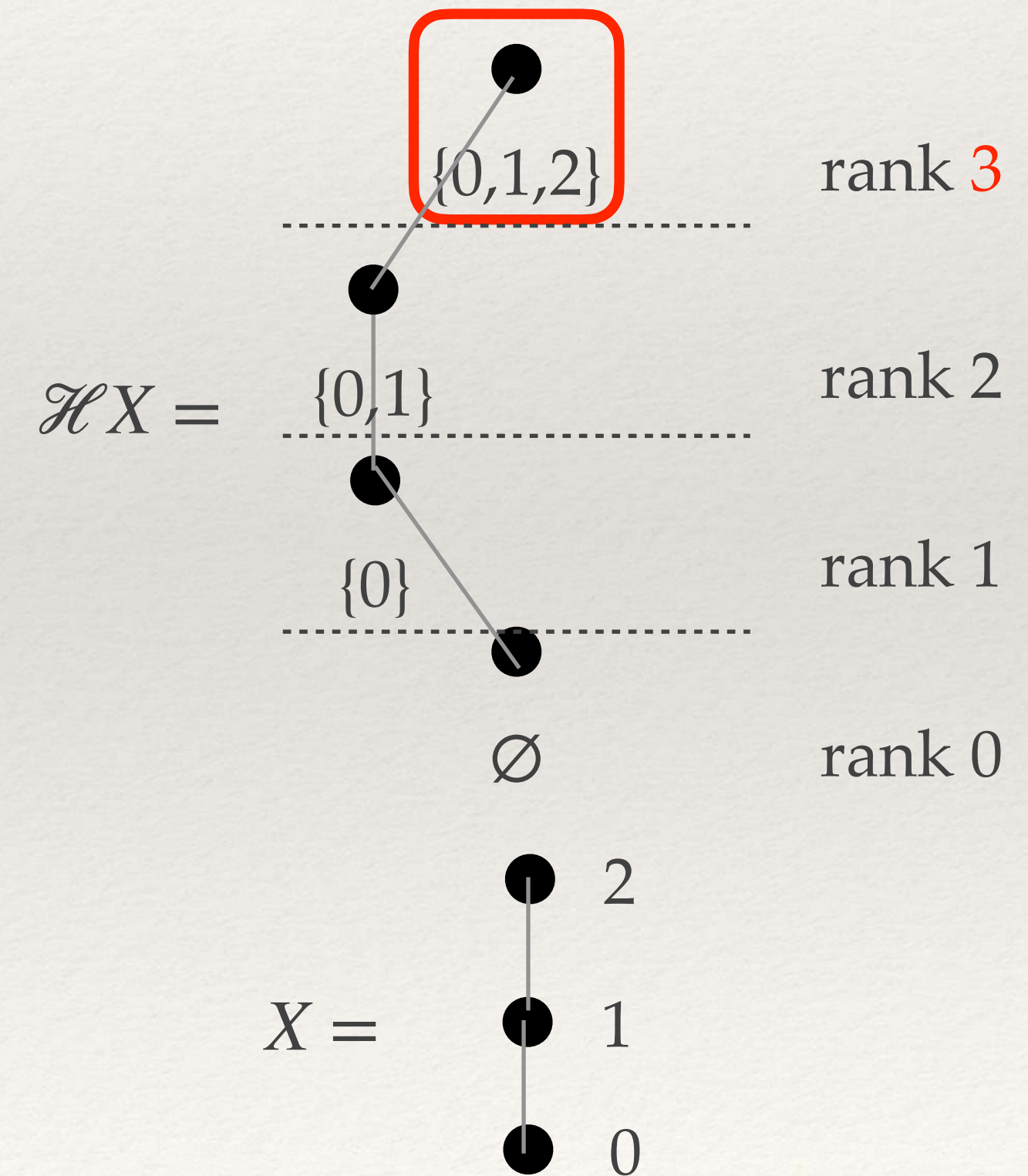
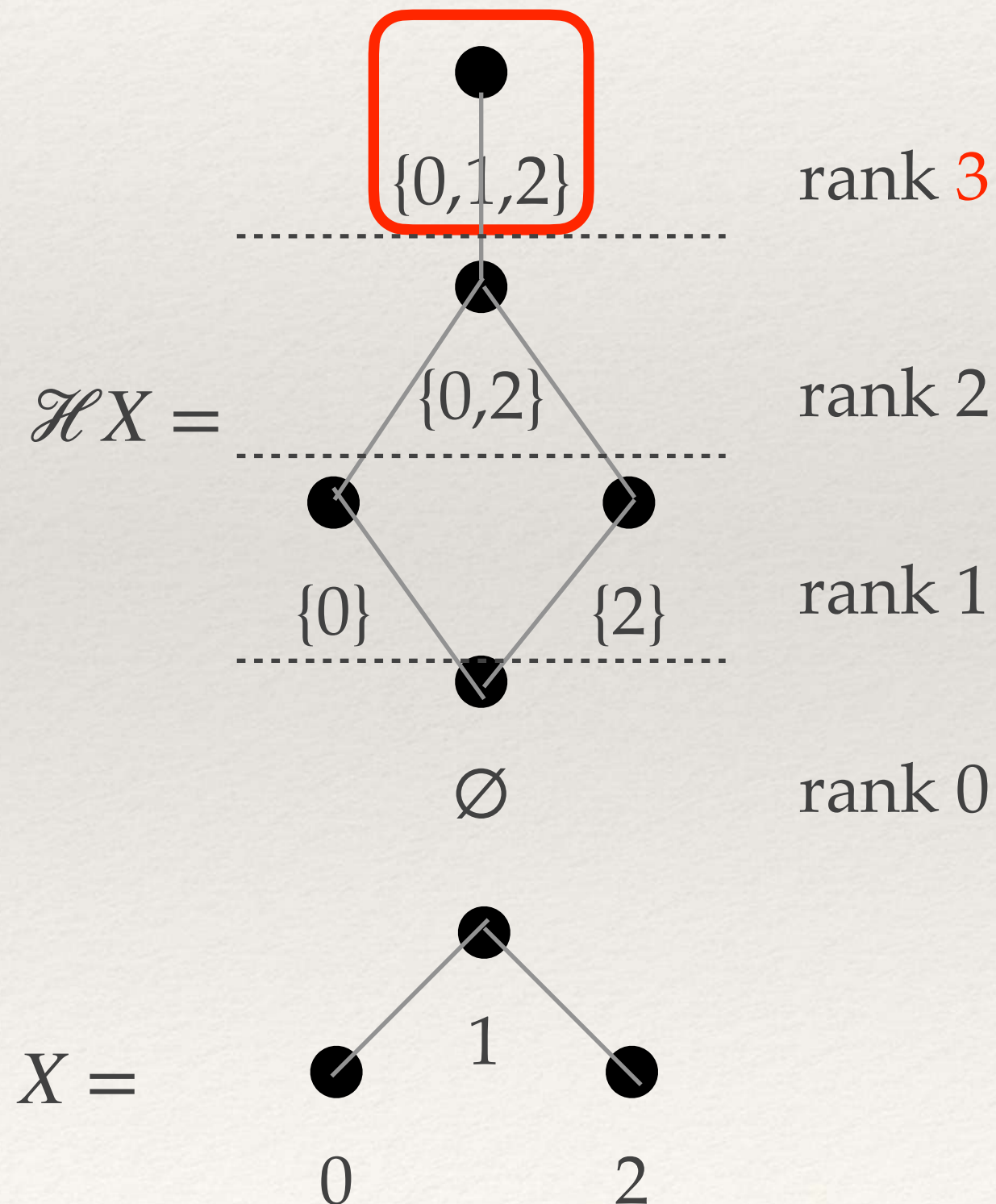
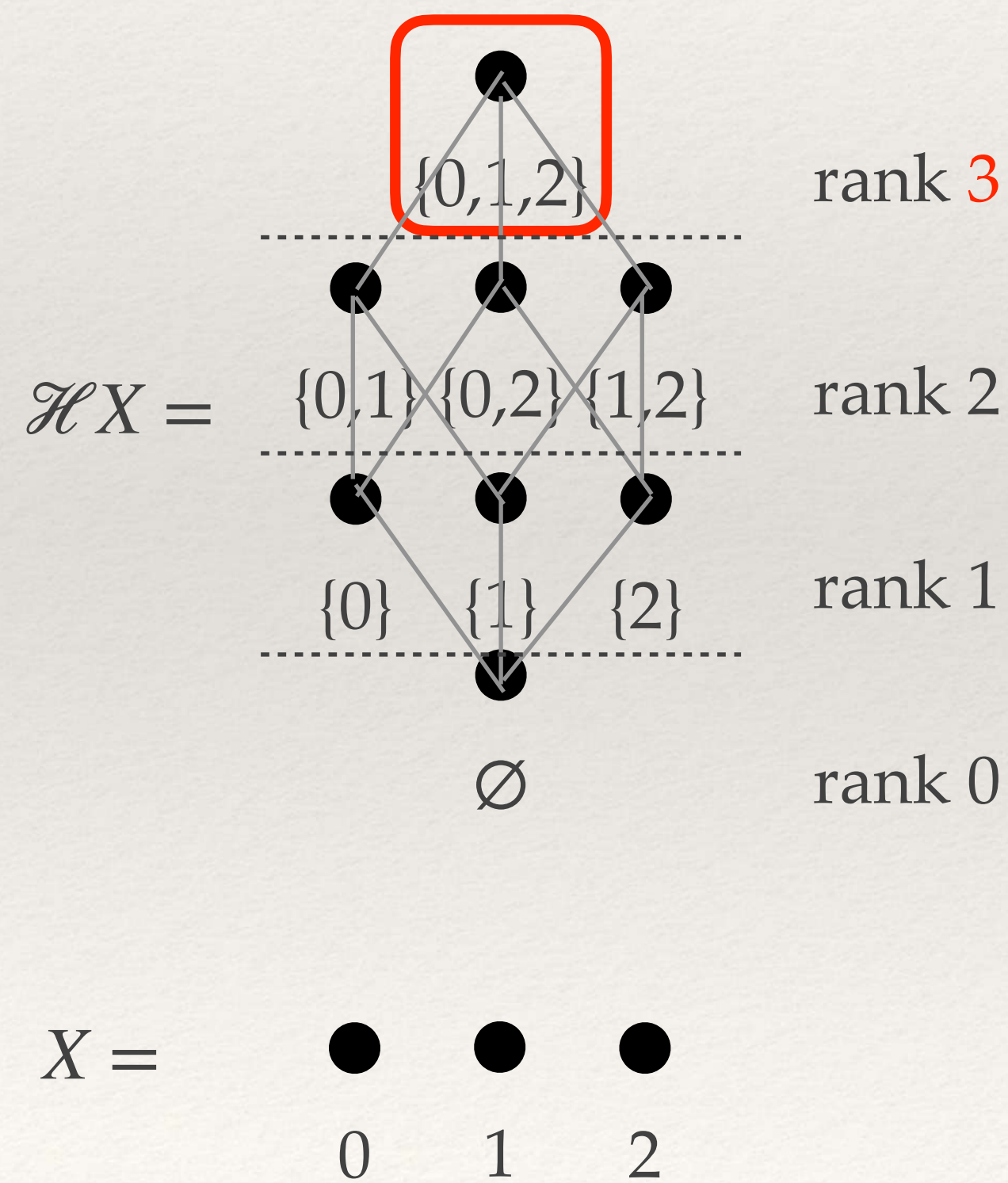
$X$  is Noetherian iff:

- (6) Every antitonic chain  $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$  of closed subsets is **stationary**
- (7)  $\mathcal{H}X$  is well-founded.



# The stature of a finite space

- For a finite  $T_0$  space  $X$ ,  $||X|| = \text{card } X$   
 ... a finite  $T_0$  space is just a finite poset, so that was known already





# The stature of $X^*$

❖ **Theorem** (JGL, Laboureix 2022). If  $X \neq \emptyset$  is Noetherian and  $\alpha = ||X||$ ,  
then  $||X^*|| = \omega^{\omega^{\alpha \pm 1}}$   
(+1 if  $\alpha = \epsilon_\beta + n$ , -1 if  $\alpha$  finite)

❖ Not very surprising: already known when  $X$  wqo (Schmidt 1979)

❖ The proof is very different, and is **localic**.

Explicitly, we do not reason on points (words),

but on **closed sets** = finite unions of word products

$(X^*)^s$  consists of **word products**

$$P ::= \epsilon \mid C^?P \mid F^*P$$

with  $C \in X^s$ ,  $F = C_1 \cup \dots \cup C_n$  ( $C_i \in X^s$ )



# An excerpt from the proof of $||X^*|| \geq \omega^{\omega^{\alpha+1}}$

- ❖ For a well-founded poset  $P$ , let  $\text{Step}(P) = \{(p, q) \mid p < q\}$ , strictly ordered by  $(p, q) < (p', q')$  iff  $q \leq p'$

**Lemma.** The ordinal height of  $\text{Step}(P)$  is that of  $P$  minus 1.

(convention: for a non-successor ordinal  $\beta$ ,  $\beta - 1 = \beta$ )

- ❖ Given  $F \in \mathcal{H}X$  and  $C \in X^s$  such that  $C \not\subseteq F$ ,  $(F, F \cup C) \in \text{Step}(\mathcal{H}X)$
- ❖ Let  $\mathbf{C}_0 = \emptyset$ ,  $\mathbf{C}_{n+1} = (F^*C^?)^n F^*$ ,  $\mathcal{A}_n = \{\mathbf{A} \in \mathcal{H}X \mid \mathbf{C}_n \subseteq \mathbf{A} \subsetneq \mathbf{C}_{n+1}\}$
- ❖ Map  $(\mathbf{B}, \mathbf{B}^+) \in \text{Step}(\mathcal{H}(F^*))$ ,  $\mathbf{A} \in \mathcal{A}_n$  to  $(F^*C^?)^{n+1} \mathbf{B} \cup \mathbf{A} C^? \mathbf{B}^+ \cup \mathbf{C}_{n+1}$
- ❖ This is **strictly monotonic** :  $\text{Step}(\mathcal{H}(F^*)) \times_{\text{lex}} \mathcal{A}_n \rightarrow \mathcal{A}_{n+1}$

A finite union  
of word products



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# An excerpt from the proof of $||X^*|| \geq \omega^{\omega^{\alpha \pm 1}}$

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- ❖ There is a **strictly monotonic** map:  $\text{Step}(\mathcal{H}(F^*)) \times_{lex} \mathcal{A}_n \rightarrow \mathcal{A}_{n+1}$
- ❖ The ordinal height of  $\mathcal{A}_{n+1}$  is  $||\mathbf{C}_{n+1}||$   
Hence if  $||F^*|| \geq \omega^{\omega^\beta}$  then  $||\mathbf{C}_{n+1}|| \geq \omega^{\omega^\alpha \times (n+1)}$ ,  
so  $||(F \cup C)^*|| \geq \omega^{\omega^{\beta+1}}$ , by taking suprema over  $n \in \mathbb{N}$
- ❖ This is the key step in a well-founded induction on  $F \in \mathcal{H}X$   
showing  $||F^*|| \geq \omega^{\omega^{||F|| \pm 1}}$
- ❖ Finally, let  $F = X$ ; by definition,  $||X|| = \alpha$ .  $\square$



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# The stature of $\mathbb{Z}[X_1, \dots, X_n]$

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- ❖ The ordinal height of the lattice of ideals of  $\mathbb{Z}[X_1, \dots, X_n]$  is  $\omega^n + 1$  (Aschenbrenner, Pong 2004)
- ❖ Hence  $||\text{Spec}(\mathbb{Z}[X_1, \dots, X_n])|| \leq \omega^n$  (and I conjecture equality—not checked out of laziness)
- ❖ Only indirectly related to wqos (through leading monomials) but remember that  $\text{Spec}(\mathbb{Z}[X_1, \dots, X_n])$  is not itself wqo.
- ❖ Together with  $||X \times Y|| = ||X|| \otimes ||Y||$  (JGL, Laboureix 2022) extending the same formula on wqos (de Jongh, Parikh 1977), we obtain the **stature** of the state space of **concurrent polynomial programs...**



# The stature of the state space of concurrent polynomial programs

- ❖  $m$  programs, each on  $n$  variables
- $p$  queues, on  $k \geq 1$  letters

- ❖ Stature of state space  $\leq$   
 $(\omega^n)^m \otimes (\omega^{\omega^{k-1}})^p$   
 $= \omega^{nm \oplus \omega^{k-1} \cdot p}$

- ❖ Note that the contribution of the polynomial programs ( $nm$ ) is **much lower** than the contribution of the queues ( $\omega^{k-1} \cdot p$ )

- ❖ What is the **actual complexity** of verifying concurrent polynomial programs?

## Concurrent polynomial programs

(JGL 2011)

- \* Finite networks of polynomial programs  $P_1, \dots, P_m$  communicating through **lossy** communication queues on a finite alphabet  $\Sigma$
- \* State space = finite **product** of
  - **spectra** of polynomial rings  $\mathbb{Z}[X_1, \dots, X_n]$ , one for each  $P_i$
  - $\Sigma^*$ , with **word topology**, one for each communication queue



# Our findings on statures so far

statures

sobrification ranks

- ❖ We have already obtained **statures** of quite a few Noetherian constructions
- ❖ We retrieve the known formulae from wqo theory, which **extend** properly
- ❖ and **new formulae** for non-wqo Noetherian spaces
- ❖ A related notion: **sobrification ranks**  $|X^s|$
- ❖ Missing: finite **trees**, notably (see Schmidt 1979 for the wqo case)

$X$	$  X  $	sob $X$
finite $T_0$	card $X$	$\leq$ card $X$
ordinal $\alpha$ (Alex.)	$\alpha$	$\alpha / \alpha+1$
$Y+Z$	$  Y   \oplus   Z  $	$\max(\text{sob } Y, \text{sob } Z)$
$Y+_{\text{lex}}Z$	$  Y   +   Z  $	$\text{sob } Y + \text{sob } Z$
$Y_{\perp}$	$1 +   Y  $	$1 + \text{sob } Y$
$Y \times Z$	$  Y   \otimes   Z  $	$(\text{sob } Y \oplus \text{sob } Z) - 1$
fin. words $Y^*$	$\omega^{\{\omega^{  Y  +1}\}}$	$\omega^{  Y  +1}$
multisets $Y^{\odot}$	$\omega^{\tilde{\alpha}} [  Y   = \alpha]$	$\omega \cdot   Y   + 1 \dots   Y   \otimes \omega + 1$
ordinal $\alpha$ (Scott)	$\alpha / \alpha - 1$	$\alpha / \alpha + 1$
cofinite topology	$\min(\text{card } Y, \omega)$	$1 / 2$
$\mathcal{H}Y, \mathbb{P}Y$	$1 +   Y   \dots \omega^{  Y  }$	$  Y   + 1$
words, prefix top.	$\omega^{\{\omega^{\beta+1}\}}$ $[  Y   = \omega^{\{\omega^{\beta+1}\}} + \dots]$	$\omega^{\alpha+1} + 1$ $[  Y   - 1 = \omega^{\alpha+1} + \dots]$
$Y < \alpha$	$\leq \omega^{\{\omega^{(  Y  +\alpha)+1}\}}$	$\leq \omega^{(  Y  +\alpha)+1}$

From JGL and B. Laboureix, *Statures and sobrification ranks of Noetherian spaces*. Submitted, 2022.  
<https://arxiv.org/abs/2112.06828>

Bottom row from JGL, S. Halfon, and A. Lopez, *Infinitary Noetherian Constructions II. Transfinite Words and the Regular Subword Topology*. Submitted, 2022.  
<https://arxiv.org/abs/2202.05047>

- ❖ Application to actual **complexity** upper bounds?



# Conclusion



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# Conclusion, research directions

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- ❖ A rich theory extending **wqos** into the topological: **Noetherian** spaces
- ❖ Old results extend, new results pop up (powersets, spectra, infinite words)
- ❖ Ordinal analysis: the **stature**  $||X||$  = ordinal rank of top element of  $\mathcal{H}X$   
as an analogue of maximal order types
- ❖ Still in its infancy