

Maximal order type
(and length of bad sequences)
for finitary powersets of WQOs

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BRAVAS TASK D: COMPLEXITY ANALYSIS

Highlights:

- ▶ The complexity analysis of the KLM and related wqo-based algorithms is at the source of recent complexity-theoretical developments.
- ▶ These developments must be consolidated and turned into a comprehensive and easy-to-use toolbox.
- ▶ Challenges: develop Length-Function Theorems for elementary wqos and beyond, refine them for powersets and multisets, apply them to adherence computations and other algorithms with ideal-based termination arguments.

COMPLEXITY FOR FINITARY POWERSSETS OF WQO

Application. Deciding trace-universality of WSTS's (including VAS's) relies on a powerset construction yielding a deterministic, trace equivalent, system. What is the complexity?

⇒ we need Length-function Theorems for $\mathcal{P}_f(\mathbb{N}^k)$, and more generally for $\mathcal{P}_f(A)$

⇒ for a start, we need to understand the m.o.t. of $\mathcal{P}_f(A)$ (def. follows)

⇒ Sylvain+I proposed an M2 internship on “Complexity for Finitary Powerset of WQOs”.

We will supervise Balasubramanian A. R. (CMI) in may–jul 2019.

Today: I'll survey the current state of knowledge of maximal-order types for finitary powersets, based on earlier unfinished work with S. Abriola (UBA).

THE MAXIMAL-ORDER TYPE OF WQOs. BASICS

Let (A, \leq) be a well-founded po. It has a natural notion of rank, or height: $h(A) \stackrel{\text{def}}{=} \text{the smallest ordinal } \alpha \text{ such that there is an embedding } f: A \rightarrow \alpha \text{ with } x < y \implies f(x) < f(y).$

In 1977, De Jongh & Parikh introduced a related notion for wqos: the maximal order-type (mot):

$$o(A, \leq) \stackrel{\text{def}}{=} \sup\{|\leq| : \leq \text{ is a well-order that extends } \leq\}$$

NB: A wqo implies that all its linearizations are well-founded

For WQOs, $o(A)$ gives better/additional information. Also: $o(A)$ is the height of bad sequences in A .

E.g., for all $k = 1, 2, \dots$, $h(\mathbb{N}^k) = \omega$ and $o(\mathbb{N}^k) = \omega^k$

BASICS

Thm [dJP] $o(A) = |\leq|$ for some linearization \leq of \leq .

Thm [dJP] $o(A \sqcup B) = o(A) \oplus o(B)$

$o(A \times B) = o(A) \otimes o(B)$

$o(A \times_{\text{lex}} B) = o(A) \cdot o(B)$

$o(\Sigma_k^*) = \omega^{\omega^{k-1}}$

Thm [dJP]

$$o(A) = \sup_{x \in A} \{o(A_{\neq x}) + 1\}$$

Thm [Schmidt, Weiermann,..]

$o(A^*) = \omega^{\omega^{o(A)-1}}$ if $0 < o(A) < \omega$

$o(\mathcal{M}_f(A)) = \omega^{\widehat{o(A)}}$

$o(T(A)) = \dots$ and the quest goes on

ORDERING POWERSSETS

For $X, Y \subseteq A$, define

$$X \sqsubseteq_H Y \stackrel{\text{def}}{\iff} \forall x \in X : \exists y \in Y : x \leq y \iff \downarrow X \subseteq \downarrow Y$$

Facts $X \equiv_H \downarrow X$

$X \equiv_H \max X$ when X is finite

$(\mathcal{P}_f(A), \sqsubseteq)$ is wqo when A is. NB: $\mathcal{P}(A)$ is in general not wqo

$(\mathcal{P}_f(A), \sqsubseteq)$ is isomorphic to $(\mathcal{A}(A), \sqsubseteq)$

Examples.

$$o(\mathcal{P}_f(\Sigma_k)) = 2^k = 2^{o(\Sigma_k)}$$

$$o(\mathcal{P}_f(\mathbb{N})) = 1 + \omega = \omega$$

TOOL: REFLECTIONS

Def. $r : A \hookrightarrow B \stackrel{\text{def}}{\iff} \forall x, y : r(x) \leq_B r(y) \implies x \leq_A y$

Fact. $A \hookrightarrow B \implies o(A) \leq o(B)$

Example. $(\mathcal{P}_f(A), \sqsubseteq) \hookrightarrow (\mathcal{M}_f(A), \sqsubseteq_m) \hookrightarrow (A^*, \leq_*)$

Fact. $A \hookrightarrow B \implies \mathcal{P}_f(A) \hookrightarrow \mathcal{P}_f(B)$

Thm. (Lower Bound)

$$o(\mathcal{P}_f(A)) \geq 1 + o(A)$$

Proof $(\Sigma_1 + A) \hookrightarrow \mathcal{P}_f(A)$

NB. If A is linear then $o(\mathcal{P}_f(A)) = 1 + o(A)$.

TOOL: SET DECOMPOSITION

Fact. $(\mathcal{P}_f(A \sqcup B), \sqsubseteq) \equiv (\mathcal{P}_f(A), \sqsubseteq) \times (\mathcal{P}_f(B), \sqsubseteq)$

Coro. $o(\mathcal{P}_f(A \sqcup B)) = o(\mathcal{P}_f(A)) \otimes o(\mathcal{P}_f(B))$

Fact. $\mathcal{P}_f(A + B) \leftrightarrow \mathcal{P}_f(A) + (\mathcal{P}_f(B) \setminus \emptyset) \leftrightarrow \mathcal{P}_f(A + B)$

Coro. $o(\mathcal{P}_f(A + B)) = o(\mathcal{P}_f(A)) + (o(\mathcal{P}_f(B)) - 1)$

Thm. (Upper Bound)

$$o(\mathcal{P}_f(A)) \leq 2^{o(A)}$$

Furthermore, $o(\mathcal{P}_f(A)) = 2^{o(A)}$ for A obtained via $\mathcal{M}_f, \sqcup, \times$.

CONCLUDING REMARKS

- ▶ $o(\mathbb{N} \cdot d) = \omega \cdot d$ and $o(\mathcal{P}(\mathbb{N} \cdot d)) = \omega^d$ with $1 + \omega \cdot d < \omega^d < 2^{\omega \cdot d} = \omega^{\omega^d}$
- ▶ Can we extend the class of WQOs (operations) for which exact values are known, perhaps by introducing extra parameters?
- ▶ Can we show Length-function theorems that are better for $\mathcal{P}_f(\mathbb{N}^k)$ than for \mathbb{N}^{k*} ?
- ▶ Complexity of universality & regular trace inclusion for VASS and extensions? [Known in dimension 1 by Hofman & Totzke]

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