

Polynomial Vector Addition Systems With States

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1 Polynomial Vector Addition Systems With States

1.1 Vector Addition Systems

VAS : a non-empty finite set $\mathbf{A} \subseteq \mathbb{Z}^d$ of actions.

Vectors in \mathbb{N}^d are the configuration.

$\mathbf{x} \rightarrow \mathbf{y}$ if $\mathbf{y} = \mathbf{x} + \mathbf{a}$ for some action $\mathbf{a} \in \mathbf{A}$.

$\xrightarrow{*}$ reflexive and transitive closure of \rightarrow

► **Example 1.** The VAS $\mathbf{A} = \{(-1, 2, 0, 0), (0, -1, 2, 0), (0, 0, -1, 2)\}$.

► **Definition 2.** A VAS is structurally bounded if for every $\mathbf{x} \in \mathbb{N}^d$ the following set is finite:

$$\{\mathbf{y} \in \mathbb{N}^d \mid \mathbf{x} \xrightarrow{*} \mathbf{y}\}$$

► **Theorem 3** (Theorem 3.10 and Theorem 3.11 of [5]). *The reachability problem for structurally bounded VAS is polynomial-space complete.*

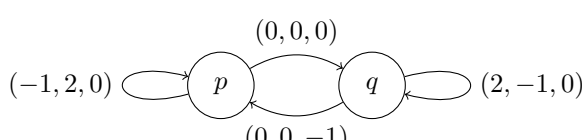
1.2 Vector Addition Systems With States

VASS : a graph $G = (Q, \mathbf{A}, E)$.

A configuration is a pair (q, \mathbf{x}) in $Q \times \mathbb{N}^d$ denoted as $q(\mathbf{x})$.

$p(\mathbf{x}) \xrightarrow{e} q(\mathbf{y})$ if $e = (p, \mathbf{y} - \mathbf{x}, q)$.

$$\begin{aligned} \xrightarrow{e_1 \dots e_k} &= \xrightarrow{e_1} \dots \xrightarrow{e_k} \\ \xrightarrow{*} &= \bigcup_{\pi \in E^*} \xrightarrow{\pi} \end{aligned}$$



► **Example 4** ([2]).

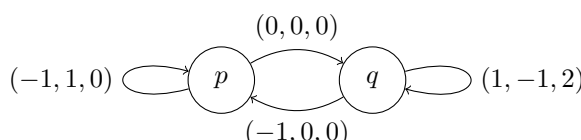
$$p(m, 0, n) \xrightarrow{(p, (-1, 2, 0), p)^m (p, (0, 0, 0), q) (q, (2, -1, 0), q)^{2m} (q, (0, 0, -1), p)} p(4m, 0, n - 1)$$

implies

$$p(1, 0, n) \xrightarrow{*} p(4^n, 0, 0)$$

1.3 Polynomial VASS

► **Definition 5.** A VASS is said to be *polynomial* if there exists a polynomial function f such that $\|\mathbf{y}\| \leq f(\|\mathbf{x}\|)$ for every $p(\mathbf{x}) \xrightarrow{*} q(\mathbf{y})$.



► **Example 6.**

$$p(m, 0, n) \xrightarrow{(p, (-1, 1, 0), p)^m (p, (0, 0, 0), q) (q, (1, -1, 2), q)^m (q, (-1, 0, 0), p)} p(m - 1, 0, n + 2m)$$

implies

$$p(n, 0, 0) \xrightarrow{*} p(0, 0, n^2 + n)$$

2 Iteration Schemes

An *iteration scheme* of a VASS G is a finite sequence $\sigma_1, \dots, \sigma_k$ of cycles such that:

$$\bigwedge_{j=1}^k \|\Delta(\sigma_j)\|^- \subseteq \|\Delta(\sigma_1) + \dots + \Delta(\sigma_k)\|^+$$

$$\|\mathbf{v}\|^+ = \{i \in \{1, \dots, d\} \mid \mathbf{v}[i] > 0\}$$

$$\|\mathbf{v}\|^- = \{i \in \{1, \dots, d\} \mid \mathbf{v}[i] < 0\}$$

■ *iterable index:* $i \in \|\Delta(\sigma_1) + \dots + \Delta(\sigma_k)\|^+$

■ *iterable edge* t occurs in $\sigma_1, \dots, \sigma_k$.

► **Lemma 7.** *For every strongly-connected VASS, there exists a rational number $\lambda > 1$ such that for every $n \in \mathbb{N}$ there exists an execution $p_n(\mathbf{x}_n) \xrightarrow{\pi_n} q_n(\mathbf{y}_n)$ such that:*

■ $\|\mathbf{x}_n\| \geq n$.

■ $\mathbf{y}_n[i] \geq \lambda^{|\mathbf{x}_n|}$ for every iterable index i .

■ Every iterable edge occurs in π_n at least $\lambda^{|\mathbf{x}_n|}$ times.

3 Computing the Set of Iterable Indexes and Edges

$(I, T) \subseteq (\{1, \dots, d\}, E)$

An (I, T) -constrained iteration scheme of a VASS G is a finite sequence $\sigma_1, \dots, \sigma_k$ of cycles in T^* such that:

$$\bigwedge_{j=1}^k \|\Delta(\sigma_j)\|^- \subseteq \|\Delta(\sigma_1) + \dots + \Delta(\sigma_k)\|^+ \subseteq I$$

We denote by $\Gamma(I, T)$ the pair (I', T') where:

■ I' indexes in $\|\Delta(\sigma_1) + \dots + \Delta(\sigma_k)\|^+$.

■ T' transitions occurring in $\sigma_1, \dots, \sigma_k$.

We want to compute $\Gamma(\{1, \dots, d\}, E)$.

À la Kosaraju-Sullivan:

Find an operator $\Omega : (I, T) \mapsto (I', T')$ such that:

■ $(I', T') \subseteq (I, T)$,

■ $\Gamma(I, T) = \Gamma(I', T')$, and

■ $\Gamma(I, T) = (I, T)$ if $(I', T') = (I, T)$.

► **Definition 8.** Let us consider the following linear system over the variables $(\mu_i)_{i \in I}$ and $(\lambda_t)_{t \in T}$ ranging over the non-negative rational numbers:

$$\begin{aligned} \bigwedge_{q \in Q} \sum_{t \in T \mid \text{tgt}(t)=q} \lambda_t &= \sum_{t \in T \mid \text{src}(t)=q} \lambda_t \\ \bigwedge_{S \in \text{SCC}(T)} \bigwedge_{i \notin I} \sum_{t \in S} \lambda_t \cdot \Delta(t)[i] &= 0 \\ \bigwedge_{i \in I} \sum_{t \in T} \lambda_t \cdot \Delta(t)[i] &= \mu_i \end{aligned}$$

Then $\Omega(I, T) = (I', T')$ where:

■ I' is the set of indexes $i \in I$ satisfying the previous linear system and $\mu_i > 0$, and

■ T' is the set of edges $t \in T$ satisfying the previous linear system and $\lambda_t > 0$.

► **Theorem 9.** *Iterable indexes and iterable edges are computable in polynomial time.*

4 Non-iterable case

► **Theorem 10.** *Let $G = (Q, \mathbf{A}, E)$ be a strongly connected VASS. For every $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$, the values $\mathbf{y}[i]$ where i is a non iterable index, and the number of occurrences of non iterable edges in π are bounded by:*

$$[(1 + \|\mathbf{x}\|)^2 d^2 (3\|\mathbf{A}\| \cdot |Q|)^{15d^4}]^{4^{d|E|}}$$

5 Applications

► **Theorem 11.** *A strongly connected VASS is polynomial if, and only if, its set of iterable indexes is empty.*

► **Lemma 12.** *A VASS is polynomial if, and only if, its restriction to every SCC is polynomial.*

► **Theorem 13.** *We can decide in polynomial-time if a VASS is polynomial.*

► **Theorem 14.** *The reachability problem for polynomial VASS is exponential-space complete.*

6 Conclusion

As a future work, we are interested in by using iteration schemes rather than iterable cycles in the KLM algorithm to hopefully obtain better complexity upper bound for the reachability problem for polynomial vector addition systems.

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