

BRAVAS

Topological FACs

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FAC

- ❖ Reminder: $wqo = FAC + \text{well-founded}$
 $FAC = \text{every antichain is finite}$
- ❖ Alain generalized WSTS to WBTS, replacing wqo by FAC
- ❖ **Théorème** [Erdős-Tarski 43, Bonnet 72, Kabil et Pouzet 92, many others]:
A poset is $FAC \Leftrightarrow$ every downset is a **finite** union of ideals.

Topological FAC?

- ❖ **Proposed definition:** a topological space X is FAC \Leftrightarrow every closed subset is a **finite** union of irreducibles.
- ❖ **Fact:** If every closed subspace of X has a dense Noetherian subspace then X is FAC.
- ❖ Converse? Non-trivial examples? Applications?

Topological FAC!

- ❖ **Theorem:** TFAE for a topological space X :
 1. X has no infinite discrete subspace
 2. Every closed subspace is a finite union of irreducibles
 3. Every closed subspace has a dense Noetherian subspace.
- ❖ $1 \Leftrightarrow 2$: [JGL, Top.Proc.2018], immediate extension of Lawson, Mislove, Priestley, *Ordered sets with no infinite antichain*, Discrete Math. 63(2):225-230, 1987.
- $1 \Leftrightarrow 2 \Leftrightarrow 3$: [JGL&M.Pouzet, soumis, 2018]

Topological antichains

- ❖ Let $A \subseteq X$. A is a (topological) **antichain** iff:
 - ❖ A is a discrete subspace
 - ❖ i.e., for every $x \in A$, x is outside $\text{cl}(A - \{x\})$
(cl = closure in X).

Quasiorders	Topological
Up sets	Opens
Down sets	Closed
Ideals	Irreducible closed

The key lemma

- ❖ **Lemma.** Let M be ∞ family of irreducible closed sets.
If: for every ∞ subfamily N of M , $\text{cl}(\cup N) = \text{cl}(\cup M)$.
Then $\text{cl}(\cup M)$ is **irreducible closed**.
- ❖ **Proof.** Let U, V be opens meeting $\text{cl}(\cup M)$ hence $\cup M$.
Let $M' = \{I \in M \mid U \text{ meets } I\}$.
If M' finite let $N = M - M'$: $\text{cl}(\cup N) = \text{cl}(\cup M)$ meets U : no.
So M' infinite, hence $\text{cl}(\cup M') = \text{cl}(\cup M)$ meets V .
Hence some I in M' meets V ; also U , hence $U \cap V$. \square

Maximal irreducibles

- ❖ **Lemma.** Let C be closed in X .
Every irreducible closed $I \subseteq C$ is included in some maximal irreducible closed $I \subseteq C$.
- ❖ **Proof.** Zorn. \square

1 \Rightarrow 2 (1/2)

- ❖ Assume no ∞ discrete subspace.
Let C be closed, not a finite union of irreducibles.
- ❖ Let $M_0 = \{\text{max. irreducible closed sets } \subseteq C\}$.
By latest Lemma, $C = \cup M_0 \dots$ hence $M_0 \infty$.
Also, $C = \text{cl}(\cup M_0)$.
By Key Lemma, find ∞ subset $M_1 / \text{cl}(\cup M_1) \subsetneq \text{cl}(\cup M_0)$.
- ❖ Iterate: $\text{cl}(\cup M_0) \supsetneq \text{cl}(\cup M_1) \supsetneq \dots \supsetneq \text{cl}(\cup M_k) \supsetneq \dots$

1 \Rightarrow 2 (2/2)

- ❖ Recall: $\text{cl}(\cup M_0) \supsetneq \text{cl}(\cup M_1) \supsetneq \dots \supsetneq \text{cl}(\cup M_k) \supsetneq \dots$
Then pick $I_k \in M_k / I_k \not\subseteq \text{cl}(\cup M_{k+1})$ for each k .
- ❖ Observe $I_k \not\subseteq \text{cl}(I_0 \cup \dots \cup I_{k-1} \cup I_{k+1} \cup \dots)$, otherwise
 $I_k \subseteq \text{cl}(I_0 \cup \dots \cup I_{k-1} \cup \text{cl}(\cup M_{k+1})) = I_0 \cup \dots \cup I_{k-1} \cup \text{cl}(\cup M_{k+1})$.
By irreducibility:
 - either $I_k \subset I_j$ for some $j < k$: no since I_k is maximal (in M_0 !)
 - or $I_k \subseteq \text{cl}(\cup M_{k+1})$: no.
- ❖ Now pick $x_k \in I_k / x_k \notin \text{cl}(I_0 \cup \dots \cup I_{k-1} \cup I_{k+1} \cup \dots)$.
Then $\{x_1, x_2, \dots, x_k, \dots\}$ is discrete and infinite: contradiction. \square