Ideally Effective WQOs

Simon Halfon

Joint work with J. Goubault-Larrecq, P. Karandikar, K. Narayan Kumar and P. Schnoebelen

LSV, ENS Cachan

BRAVAS
Motivation

- Up-sets:

\[ U = F_1 \cup F_2 \cup \cdots \cup F_n \]
Motivation

- Up-sets:

\[ U = F_1 \cup F_2 \cup \cdots \cup F_n \]

\( WQO \Rightarrow \) filters = principal filters = \( \uparrow x \)

- Down-sets:

\[ D = I_1 \cup I_2 \cup \cdots \cup I_n \]

Principal ideals are ideals: \( \downarrow x \), but not only.

\[ U = \uparrow(1, 2) \cup \uparrow(3, 1) \]
Motivation

- **Up-sets:**

  \[ U = F_1 \cup F_2 \cup \cdots \cup F_n \]

  WQO $\Rightarrow$ filters = principal
  filters $=$ $\uparrow x$

- **Down-sets:**

  \[ D = I_1 \cup I_2 \cup \cdots \cup I_n \]

\[ U = \uparrow(1, 2) \cup \uparrow(3, 1) \]
Motivation

- **Up-sets:**
  \[ U = F_1 \cup F_2 \cup \cdots \cup F_n \]
  
  \[ \text{WQO} \Rightarrow \text{filters} = \text{principal filters} = \uparrow x \]

- **Down-sets:**
  \[ D = I_1 \cup I_2 \cup \cdots \cup I_n \]
  
  Principal ideals are ideals: \( \downarrow x \), but not only.

\[ U = \uparrow(1, 2) \cup \uparrow(3, 1) \]

\[ D = \downarrow(2, \omega) \cup \downarrow(3, 2) \]
A WQO \((X, \leq)\) is Ideally Effective if:

- \((X, \leq)\) is effective
- \((Idl(X), \subseteq)\) is effective
- Complements and Intersections of filters and ideals are computable
Definition

A WQO $(X, \leq)$ is Ideally Effective if:

- $(X, \leq)$ is effective
- $(\text{Idl}(X), \subseteq)$ is effective
- Complements and Intersections of filters and ideals are computable

- Up-sets = lists of elements of $X$. 
Ideally Effective WQO

**Definition**

A WQO \((X, \leq)\) is Ideally Effective if:

- \((X, \leq)\) is effective
- \((Idl(X), \subseteq)\) is effective
- Complements and Intersections of filters and ideals are computable

- Up-sets = lists of elements of \(X\).
- Down-sets = lists of ideals.
Ideally Effective WQO

**Definition**

A WQO \((X, \leq)\) is Ideally Effective if:

- \((X, \leq)\) is effective
- \((Idl(X), \subseteq)\) is effective
- Complements and Intersections of filters and ideals are computable

- Up-sets = lists of elements of \(X\).
- Down-sets = lists of ideals.
- Compute minimal representations:
  \[\uparrow(1, 2) \cup \uparrow(2, 1) \cup \uparrow(2, 2) = \uparrow(1, 2) \cup \uparrow(2, 1)\]
- Lift all set-theoretic operations to Up- and Down-sets.
(VASS):

\[ \mathbb{N}^d \Rightarrow \omega \text{-vectors} \]
Examples

(VASS): \( \mathbb{N}^d \Rightarrow \omega \)-vectors

(Channel Systems): \( A^* \Rightarrow \text{SRE} \)

(Universality Problem):
\[
P f(\mathbb{N}^d) \Rightarrow P f(\text{Down}(\mathbb{N}^d)) \equiv P f(A(\mathbb{N}^d \omega))
\]

(Timed Petri Nets):
\[
(\mathbb{N}^d)^* \Rightarrow \text{SRE of } \omega \text{-vectors}
\]

(Production System):
\[
0^* \cdots \ast \Rightarrow \text{SRE of SRE}
\]
Examples

(VASS): $\mathbb{N}^d \Rightarrow \omega$-vectors

(Channel Systems): $A^* \Rightarrow$ SRE

(Universality Problem): $\mathcal{P}_f(\mathbb{N}^d) \Rightarrow \mathcal{P}_f(\text{Down}(\mathbb{N}^d)) \equiv \mathcal{P}_f(A(\mathbb{N}_\omega^d))$
Examples

(VASS): $\mathbb{N}^d \Rightarrow \omega$-vectors

(Channel Systems): $A^* \Rightarrow$ SRE

(Universality Problem): $\mathcal{P}_f(\mathbb{N}^d) \Rightarrow \mathcal{P}_f(\text{Down}(\mathbb{N}^d)) \equiv \mathcal{P}_f(A(\mathbb{N}^d))$

(Timed Petri Nets): $(\mathbb{N}^d)^* \Rightarrow$ SRE of $\omega$ vectors
Examples

(VASS): \( \mathbb{N}^d \) \( \Rightarrow \) \( \omega \)-vectors

(Channel Systems): \( A^* \) \( \Rightarrow \) SRE

(Universality Problem): \( \mathcal{P}_f(\mathbb{N}^d) \) \( \Rightarrow \) \( \mathcal{P}_f(\text{Down}(\mathbb{N}^d)) \equiv \mathcal{P}_f(A(\mathbb{N}_\omega^d)) \)

(Timed Petri Nets): \( (\mathbb{N}^d)^* \) \( \Rightarrow \) SRE of \( \omega \) vectors

(Priority Channel): \( \{0^* \}^d \) \( \Rightarrow \) SRE of SRE
Examples

(VASS): \( \mathbb{N}^d \Rightarrow \omega \text{-vectors} \)

(Channel Systems): \( A^* \Rightarrow \text{SRE} \)

(Universality Problem): \( \mathcal{P}_f(\mathbb{N}^d) \Rightarrow \mathcal{P}_f(\text{Down}(\mathbb{N}^d)) \equiv \mathcal{P}_f(A(\mathbb{N}_\omega^d)) \)

(Timed Petri Nets): \( (\mathbb{N}^d)^* \Rightarrow \text{SRE of } \omega \text{ vectors} \)

(Priority Channel): \( \{ \cdots \}^d \Rightarrow \text{SRE of } \text{SRE} \)

(Universality Problem): \( \text{Down}(\mathbb{N}^d) \equiv A(\mathbb{N}_\omega^d) \Rightarrow A(A(\omega + 2)) \)
“Natural” WQOs are ideally effective

Base Cases:
Finite orderings and Linear orderings

Inductive Cases:
($(X, \leq_X), (Y, \leq_Y)$ ideally effective WQO. The following are ideally effective:

- $(X \times Y, \leq_X \times \leq_Y)$
- $(X^*, \leq*)$
- $(\mathcal{P}_f(X), \leq_H)$
- $(X^\otimes, \leq_{emb})$
- $(X^\ast, \leq_{ms})$
- $(X \times Y, \leq_{lex})$
- $(X \sqcup Y, \leq_X \sqcup \leq_Y)$
- $(X \oplus Y, \leq_X \oplus \leq_Y)$

And algorithms for set-theoretic operations can be computed from algorithms for set-theoretic operations in $X$ and $Y$ (OCaml Functor).
“Semi-natural” WQOs are ideally effective as well

$(X, \leq)$ is an ideally effective WQO:

- $\leq \subseteq \leq' \Rightarrow (X, \leq')$ is a WQO.
  Under extra assumptions, it is ideally effective.
“Semi-natural” WQOs are ideally effective as well

\( (X, \leq) \) is an ideally effective WQO:

- \( \leq \subseteq \leq \Rightarrow (X, \leq) \) is a WQO.
  Under extra assumptions, it is ideally effective.
- Particular case: Quotients.
  E.g. \( (X^\ast, \leq_{\text{emb}}) = (X^\ast / \sim, \leq_\ast \circ \sim) \).
“Semi-natural” WQOs are ideally effective as well

\((X, \leq)\) is an ideally effective WQO:

- \(\leq \subseteq \leq' \Rightarrow (X, \leq')\) is a WQO.
  Under extra assumptions, it is ideally effective.

- Particular case: Quotients.
  E.g. \((X^\ast, \leq_{\text{emb}}) = (X^\ast / \simeq, \leq_{\ast} \circ \simeq)\).

- \(Y \subseteq X \Rightarrow (Y, \leq)\) is a WQO.
  Under extra assumptions, it is ideally effective.

Applications:
- Words with stuttering:
  \(aaddaaabba \leq_{\text{st}} \text{abracadabra}\)
- Words on a circle:
  \(baaada \leq_{\text{st}} \text{abracadabra}\)
“Semi-natural” WQOs are ideally effective as well

\((X, \leq)\) is an ideally effective WQO:

- \(\leq \subseteq \leq' \Rightarrow (X, \leq')\) is a WQO. Under extra assumptions, it is ideally effective.

- Particular case: Quotients. E.g. \((X^\ast, \leq_{\text{emb}}) = (X^* / \sim, \leq^* \circ \sim)\).

- \(Y \subseteq X \Rightarrow (Y, \leq)\) is a WQO. Under extra assumptions, it is ideally effective.

Applications:

- Words with stuttering: \(aaddaaabba \leq_{\text{st}} abracadabra\)

- Words on a circle: \(baaada \leq_{\text{st}} abracadabra\)
Inclusion is always decidable in polynomial time.
Sums, Products, Finite Powerset preserve polynomial time.
Finite Sequences and Multisets don’t.
Conclusion

- Effective toolbox for manipulating up/down sets of many WQOs, encountered in particular in verification
- Next step: Trees and Graphs.
- Effective FAC?