Open Boundedness and Coverability Problems for Pushdown Vector Addition Systems

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Pushdown Vector Addition Systems

... are products of VAS with pushdown automata

\[ push(A), \begin{pmatrix} -1 \\ 0 \end{pmatrix} \]

\[ pop(A), \begin{pmatrix} 2 \\ 0 \end{pmatrix} \]

\[ nop, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \]
... are products of VAS with pushdown automata

\[
p, \bot, \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]

\[
push(A), \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]

\[
\text{nop}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\]

\[
pop(A), \begin{pmatrix} 2 \\ 0 \end{pmatrix}
\]
Pushdown Vector Addition Systems

... are products of VAS with pushdown automata

$\text{push}(A), \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\text{nop}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\text{pop}(A), \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$p, \bot, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow p, A\bot, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
Pushdown Vector Addition Systems

... are products of VAS with pushdown automata

\[ \text{push}(A), \begin{pmatrix} -1 \\ 0 \end{pmatrix} \]

\[ \text{pop}(A), \begin{pmatrix} 2 \\ 0 \end{pmatrix} \]

\[ \text{nop}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \]

\[ p, \bot, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow p, AA\bot, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow q, AA\bot, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
Pushdown Vector Addition Systems

... are products of VAS with pushdown automata

push(A), \begin{pmatrix} -1 \\ 0 \end{pmatrix}

nop, \begin{pmatrix} 0 \\ -1 \end{pmatrix}

pop(A), \begin{pmatrix} 2 \\ 0 \end{pmatrix}

p, \perp, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow p, AA \perp, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow q, AA \perp, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow q, \perp, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \rightarrow q, \perp, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow q, \perp, \begin{pmatrix} 4 \\ 0 \end{pmatrix}
... are products of VAS with pushdown automata

They can for example model recursive programs with variables over $\mathbb{N}$

1: $x \leftarrow n$
2: procedure DoubleX
3: if ($\star \land x > 0$) then
4: $x \leftarrow (x - 1)$
5: DoubleX
6: end if
7: $x \leftarrow (x + 2)$
8: end procedure
Pushdown Vector Addition Systems

... are products of VAS with pushdown automata

- They can for example model recursive programs with variables over $\mathbb{N}$

- They can also model concurrent systems with (limited) recursion
  - One recursive server + unboundedly many finite-state clients

- Is the model too powerful?
Brief non-Exhaustive State of the Art

- **Reachability**: does \((p, \varepsilon, u) \xrightarrow{*} (q, \varepsilon, v)\) ?

- **Coverability**: does there exist \(v' \geq v\) with \((p, \varepsilon, u) \xrightarrow{*} (q, \varepsilon, v')\) ?

- **Boundedness**: is \(\{(q, \sigma, v) \mid (p, \varepsilon, u) \xrightarrow{*} (q, \sigma, v)\}\) finite ?

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<td>EXPSPACE-(c_1)</td>
<td>EXPSPACE-(c_1)</td>
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<td>+ stack</td>
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<td>TOWER-(h_5), ?</td>
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[1] Lipton 1976 ; Rackoff 1978
    Leroux, Schmitz 2015
Reachability Tree of a Pushdown VAS

- Exhaustive and enumerative forward exploration from \((q_{\text{init}}, \varepsilon, v_{\text{init}})\)
  - Potentially infinite, need to truncate

\[
q_{\text{init}}, \varepsilon, v_{\text{init}} \quad \downarrow \quad q, \sigma, v \\
\downarrow \\
q_1, \sigma_1, v_1 \quad q_n, \sigma_n, v_n
\]
Tentative Simulation-Based Truncation for Pushdown VAS

Truncation Rule

\[ q_{\text{init}}, \varepsilon, v_{\text{init}} \]

\[ q, \sigma, v \]

\[ q', \sigma', v' \]

if \( q = q', v \leq v' \) and \( \sigma \leq_{\text{prefix}} \sigma' \)

\( \therefore \) No loss of information to decide boundedness

But...
Tentative Simulation-Based Truncation for Pushdown VAS

**Truncation Rule**

\[ q_{\text{init}}, \varepsilon, v_{\text{init}} \]

\[ q, \sigma, v \]

\[ q', \sigma', v' \]

if \( q = q' \), \( v \leq v' \) and \( \sigma \leq_{\text{prefix}} \sigma' \)

\( \Rightarrow \) No loss of information to decide boundedness

But...

The reduced reachability tree may be infinite!
Reduced Reachability Tree for Pushdown VAS

**Truncation Rule**

\[ q_{\text{init}}, \varepsilon, v_{\text{init}} \]

\[ q, \sigma, v \]

\[ (_{\_}, _{\_}, \rho) \]

\[ q', \sigma', v' \]

If \( q = q' \) and \( v \leq v' \)

\[ \sigma \leq_{\text{suffix}} \rho \text{ for all } \rho \]
Reduced Reachability Tree for Pushdown VAS

**Truncation Rule**

\[
\begin{aligned}
q_{\text{init}}, & \varepsilon, v_{\text{init}} \\
q, & \sigma, v \\
(_&, & _&, \rho) \\
q', & \sigma', v'
\end{aligned}
\]

if \( q = q' \) and \( v \leq v' \)

\( \sigma \leq \text{suffix } \rho \) for all \( \rho \)

The reduced reachability tree

- is computable and **finite**
- contains enough information to decide boundedness
- has at most an **hyper-Ackermannian** number of nodes (the bound is tight)

**Theorem ([Leroux, Praveen, S. 2014])**

*Boundedness is decidable for pushdown VAS*
Open Boundedness Problems

**Boundedness Problem for Pushdown VAS**

- Lower bound: tower of exponentials ($F_3$) from [Lazić 2012]
- Upper bound: hyper-Ackermann ($F_{\omega^\omega}$)

Refinements: stack-boundedness and counters-boundedness

**Counters-Boundedness Problem for Pushdown VAS**

- Decidability: open
- Dim. one: NP-hard, \textsc{ExpTime}-easy [Leroux, Sutre, Totzke 2015]

Location-specific boundedness is at least as hard as reachability
Observation ([Lazić 2012])

Reachability in dimension $d$ reduces to Coverability in dimension $d + 1$.

**Proof.** Budget construction. Use the stack to test the budget for zero. Add a new counter $B$ and two new stack symbols $A, \$$. 

\[
(q_{\text{init}}^A, \varepsilon, 0) \xrightarrow{*} (q_{\text{final}}^A, \varepsilon, 0) \quad \text{iff} \quad (q_{\text{init}}^{A'}, \varepsilon, 0, 0) \xrightarrow{*} (q_{\text{final}}^{A'}, \varepsilon, \_ , \_ )
\]
Observation ([Lazić 2012])
Reachability in dimension $d$ reduces to Coverability in dimension $d + 1$.

$\text{Reach}(0) \sqsubseteq \text{Cover}(1) \sqsubseteq \text{Reach}(1) \sqsubseteq \text{Cover}(2) \sqsubseteq \cdots$
Coverability versus Reachability in Pushdown VAS

Observation ([Lazić 2012])
Reachability in dimension $d$ reduces to Coverability in dimension $d + 1$.

\[
\text{Reach}(0) \subseteq \text{Cover}(1) \subseteq \text{Reach}(1) \subseteq \text{Cover}(2) \subseteq \cdots
\]

Theorem ([Leroux, Sutre, Totzke 2015])
Coverability for 1-dimensional pushdown VAS is decidable.
Open Coverability/Reachability Problems

Coverability Problem for Pushdown VAS

- Lower bound: tower of exponentials ($F_3$) from [Lazić 2012]
- Decidability: open

Coverability for Pushdown VAS in Dimension One

- $PSPACE$-hard by reduction from bounded OCA [Lazić et al.]
- $EXPSPACE$-easy [Leroux, Sutre, Totzke 2015]

Reachability is open even in dimension one